

# Double Lie algebras, semidirect product, and integrable systems

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## Abstract

We study integrable systems on double Lie algebras in absence of Ad-invariant bilinear form by passing to the semidirect product with the  $\tau$ -representation. We show that in this stage a natural Ad-invariant bilinear form does exist, allowing for a straightforward application of the AKS theory, and giving rise to Manin triple structure, thus bringing the problem to the realm of Lie bialgebras and Poisson-Lie groups.

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# 1 Introduction

The deep relation between integrable systems and Lie algebras finds its optimal realization when the involved Lie algebra is equipped with an ad-invariant non-degenerate symmetric bilinear form. There, the coadjoint orbit setting turns to be equivalent to the Lax pair formulation, the Adler-Kostant-Symes theory of integrability [1],[6],[13] works perfectly, the Poisson-Lie group structures and Lie bialgebras naturally arises, etc. Semisimplicity is an usual requirement warranting a framework with this kind of bilinear form, however out of this framework it becomes in a rather stringent condition. This is the case with semidirect product Lie algebras.

Integrable systems can also be modelled on Lie groups, and their Hamiltonian version is realized on their cotangent bundle. There, reduction of the cotangent bundle of a Lie group by the action of some Lie subgroup brings the problem to the realm of semidirect products [5], [8] where, in spite of the lack of semisimplicity, an ad-invariant form can be defined provided the original Lie algebra had one. Also, at the Lie algebra level, in ref. [14] the complete integrability of the Euler equations on a semidirect product of a semisimple Lie algebra with its adjoint representation was proven. In ref. [3], the AKS theory was applied to study integrable systems on this kind of Lie groups.

However, the lack of an ad-invariant bilinear form is not an obstruction to the application of AKS ideas. In fact, in ref. [10],[11] the AKS theory is adapted to a context equipped with a symmetric and nondegenerate bilinear form, which also produces a decomposition of the Lie algebra into two complementary orthogonal subspaces. This is performed by using the *B operation* introduced by Arnold in [2], and realizing that it amounts to be an action of the Lie algebra on itself, which can be promoted to an action of the Lie group on its Lie algebra, called the  $\tau$ -action. Thus, the restriction of the system to one of its orthogonal components becomes integrable by factorization.

The main goal of this work is to study integrable systems on a semidirect product of a Lie algebra with its adjoint representation, disregarding the ad-

invariance property of the bilinear form. So, the framework is that of a double Lie algebra  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  equipped with a symmetric nondegenerate bilinear form, and the semidirect product  $\mathfrak{h} = \mathfrak{g} \ltimes_{\tau} \mathfrak{g}$  where the left  $\mathfrak{g}$  act on the other one by the  $\tau$ -action. The main achievement is the introduction of an  $\text{ad}^{\mathfrak{h}}$ -invariant symmetric nondegenerate bilinear form which induces a decomposition  $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ , with  $\mathfrak{h}_+, \mathfrak{h}_-$  being Lie subalgebras and isotropic subspaces. In this way, a natural Manin triple structure arises on the Lie algebra  $\mathfrak{h}$ , bringing the problem into the realm of the original AKS theory and in that of Lie bialgebras and Poisson-Lie groups. In fact, we show in which way integrable systems on  $\mathfrak{h}_{\pm}$  arising from the restriction of an almost trivial system on  $\mathfrak{h}$  defined by an  $\text{ad}^{\mathfrak{h}}$ -invariant Hamilton function, can be solved by the factorization of an exponential curve in the associated connected simply-connected Lie group  $H$ . Moreover, we built explicitly the Poisson-Lie structures on the factors  $H_{\pm}$  of the group  $H$ .

As the application of main interest, we think of that derived from Lie group with no bi-invariant Riemannian metric. From the result by Milnor [9] asserting that a Riemannian metric is bi-invariant if and only if the Lie group is a product of a compact semisimple and Abelian groups, one finds a wide class of examples fitting in the above scheme among the solvable and nilpotent Lie algebras. Many examples with dimension up to 6 are studied in ref. [4], one of them is fully developed in the present work as an example.

The work is ordered as follows: in Section II we fix the algebraic tools of the problem by introducing the  $\tau$ -action, in Section III we present the main results of this work, dealing with many issues in the semidirect product framework. In Section IV we show how works the integrability by factorization in the framework developed in the previous section. In Section V, we present three examples without Ad-invariant bilinear forms on which we apply the construction developed in the previous sections. Finally, in Section VI we include some conclusions.

## 2 Double Lie algebras and the $\tau$ -action

Let us consider a *double Lie group*  $(G, G_+, G_-)$  and its associated *double Lie algebra*  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ . These mean that  $G_+$  and  $G_-$  are Lie subgroups of  $G$  such that  $G = G_+ G_-$ , and that  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  are Lie subalgebras of  $\mathfrak{g}$  with  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ . We also assume there is a symmetric nondegenerate bilinear form  $(\cdot, \cdot)_{\mathfrak{g}}$  on  $\mathfrak{g}$ , which induces the direct sum decomposition:

$$\mathfrak{g} = \mathfrak{g}_+^{\perp} \oplus \mathfrak{g}_-^{\perp}$$

where  $\mathfrak{g}_{\pm}^{\perp}$  are the annihilators subspaces of  $\mathfrak{g}_{\pm}$ , respectively,

$$\mathfrak{g}_{\pm}^{\perp} := \left\{ Z \in \mathfrak{g} : (Z, X)_{\mathfrak{g}} = 0 \quad \forall X \in \mathfrak{g}_{\pm} \right\}$$

Since the bilinear form is not assumed to be  $\text{Ad}^G$ -invariant, the adjoint action is not a good symmetry in building integrable systems. Following references [10], [11], where AKS ideas are adapted to a framework lacking of an ad-invariant bilinear form by using the so called  $\tau$ -action, we take this symmetry as the building block of our construction. Let us briefly review the main result of these references: let  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  as above, then the adjoint action induces the  $\tau$ -action defined as

$$\text{ad}^{\tau} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}) \quad / \quad (\text{ad}_X^{\tau} Z, Y)_{\mathfrak{g}} := -(Z, [X, Y])_{\mathfrak{g}} \quad (1)$$

$\forall X, Y, Z \in \mathfrak{g}$ . It can be promoted to an action of the associated Lie group  $G$  on  $\mathfrak{g}$  through the exponential map, thereby

$$\tau : G \rightarrow \text{Aut}(\mathfrak{g}) \quad / \quad (\tau(g) X, Y)_{\mathfrak{g}} := (X, \text{Ad}_{g^{-1}}^G Y)_{\mathfrak{g}}$$

Often we also use the notation  $\tau_g X = \tau(g) X$ .

It is worth to observe that, since the bilinear form is nondegenerate, it allows for the identification of the  $\mathfrak{g}$  with its dual vector space  $\mathfrak{g}^*$  through the isomorphism

$$\psi : \mathfrak{g} \longrightarrow \mathfrak{g}^* \quad / \quad \langle \psi(X), Y \rangle := (X, Y)_{\mathfrak{g}}$$

which connects the  $\tau$ -action with the *coadjoint* one:

$$\tau(g) = \bar{\psi} \circ \text{Ad}_{g^{-1}}^{G*} \circ \psi$$

It reduces to the adjoint action, namely  $\tau(g) = \text{Ad}_g^G$ , when the bilinear form is Ad-invariant. Also,  $\psi$  provides the isomorphisms  $\psi : \mathfrak{g}_\pm^\perp \longrightarrow \mathfrak{g}_\mp^*$ . Let  $\Pi_{\mathfrak{g}_\pm^\perp} : \mathfrak{g} \rightarrow \mathfrak{g}_\pm^\perp$  be the projection map.

Observe that the annihilator subspaces  $\mathfrak{g}_\pm^\perp$  can be regarded as  $\tau$ -representation spaces of  $G_\pm$ , respectively. Moreover, the  $\tau$ -action gives rise to crossed actions, as explained in the following lemma.

**Lemma:** *The maps  $\tilde{\tau} : G_\mp \times \mathfrak{g}_\pm^\perp \rightarrow \mathfrak{g}_\pm^\perp$  defined as*

$$\tilde{\tau}(h_\mp) Z_\pm^\perp = \Pi_{\mathfrak{g}_\pm^\perp}(\tau(h_\mp) Z_\pm^\perp) \quad (2)$$

*are left actions, and the infinitesimal generator associated to elements  $Y_\mp \in \mathfrak{g}_\mp$  is*

$$(Y_\mp)_{\mathfrak{g}_\pm^\perp}(Z_\pm^\perp) = \Pi_{\mathfrak{g}_\pm^\perp}(\text{ad}_{Y_\mp}^\tau Z_\pm^\perp)$$

Thus,  $\mathfrak{g}_\pm^\perp$  is a  $G_\mp$ -space through the  $\text{Ad}^{G^*}$ -action translated via the identification  $\mathfrak{g}_\pm^\perp \simeq \mathfrak{g}_\mp^*$  induced by the inner product. Further, this last identification allows to consider the annihilators as Poisson manifolds, and the orbits of the  $G_\mp$  on  $\mathfrak{g}_\pm^\perp$  as symplectic manifolds. Under favorable circumstances (i.e. when  $f \in C^\infty(\mathfrak{g})$  is  $\text{Ad}^G$ -invariant) the dynamical system defined on these symplectic spaces by the restriction of  $f$  can be solved through the action of the other group. As a bonus, the curve on this group whose action produces the solution can be obtained by factorization of a simpler curve. Below we will see the way in which it can be generalized to a context where there are no Ad-invariant inner product.

**Remark:** *The assignment  $\text{ad}^\tau : \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g})$  is a Lie algebra antihomomorphism*

$$\text{ad}_{[X,Y]}^\tau = -[\text{ad}_X^\tau, \text{ad}_Y^\tau]$$

**Remark:** [10] *The  $\tilde{\tau}$ -orbits  $\mathcal{O}_X^\tau := \{\tilde{\tau}(h_\mp) X \in \mathfrak{g}_\pm^\perp / h_\mp \in G_\mp\}$  whose tangent space are*

$$T_{\tau(g)X} \mathcal{O}_X^\tau = \{\text{ad}_Y^\tau \tau_i(g) X / Y \in \mathfrak{g}\}$$

are symplectic manifold with the symplectic form

$$\langle \omega, \text{ad}_Y^\tau \tau(g) X \otimes \text{ad}_Z^\tau \tau(g) X \rangle_{\tau_i(g)X} := (\tau(g) X, [Y, Z])_{\mathfrak{g}}$$

### 3 Semidirect product with the $\tau$ -action representation

The lack of an Ad-invariant bilinear form on  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  is not an obstacle to have AKS integrable systems in the *double Lie algebra*  $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$  [10]. However, it prevents the existence of richer structures as the *Manin triple* one, which brings it to the realm of *Lie bialgebras* and the associated Lie groups acquire a natural compatible Poisson structure turning them into *Poisson-Lie groups*.

In this section we introduce the semidirect sum Lie algebra  $\mathfrak{h} = \mathfrak{g} \ltimes_{\tau} \mathfrak{g}$  as the central object of our construction.

#### 3.1 Lie algebra structure on $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}$ and ad-invariant bilinear form

Let us consider the semidirect sum Lie algebra  $\mathfrak{h} = \mathfrak{g} \ltimes_{\tau} \mathfrak{g}$  where the first component acts on the second one through the  $\text{ad}^\tau$ -action (1), giving rise to the Lie bracket on  $\mathfrak{h}$

$$[(X, Y), (X', Y')] := ([X, X'], \text{ad}_X^\tau Y' - \text{ad}_{X'}^\tau Y) \quad (3)$$

Also, we equip  $\mathfrak{h}$  with the symmetric nondegenerate bilinear form  $(-, -)_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{R}$

$$((X, Y), (X', Y'))_{\mathfrak{h}} := (X, Y')_{\mathfrak{g}} + (Y, X')_{\mathfrak{g}} \quad (4)$$

It is easy to prove the next result.

**Proposition:** *The bilinear form  $(\cdot, \cdot)_{\mathfrak{h}}$  on the semidirect sum of Lie algebras  $\mathfrak{h}$  is  $\text{ad}^{\mathfrak{h}}$ -invariant*

$$([(X'', Y''), (X, Y)], (X', Y'))_{\mathfrak{h}} + ((X, Y), [(X'', Y''), (X', Y')])_{\mathfrak{h}} = 0$$

Let us denote by  $H = G \ltimes_{\tau} \mathfrak{g}$  the Lie group associated with the above Lie algebra structure. Indeed, there are two possible semidirect product Lie group structures related to the left or right character of the action of  $G$  on  $\mathfrak{g}$ . We adopt the right action structure for the Lie group structure in  $H = G \ltimes \mathfrak{g}$

$$(g, X) \cdot (k, Y) := (gk, \tau_{k^{-1}} X + Y) \quad (5)$$

The exponential map in this case is

$$\text{Exp}(t(X, Y)) = \left( e^{tX}, - \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} t^n (\text{ad}_X^{\tau})^{n-1} \right) Y \right) \quad (6)$$

and the adjoint action of  $H$  on  $\mathfrak{h}$  is

$$\text{Ad}_{(g, Z)}^H(X, Y) = \left( \text{Ad}_g^G X, \tau_g(Y - \text{ad}_X^{\tau} Z) \right)$$

while the adjoint action of  $\mathfrak{h}$  on itself retrieves the Lie bracket structure (3).

### 3.2 Manin triple and factorization

The direct sum decompositions of  $\mathfrak{g}$  as  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  and  $\mathfrak{g} = \mathfrak{g}_+^{\perp} \oplus \mathfrak{g}_-^{\perp}$  allows to decompose the Lie algebra  $\mathfrak{h}$  as a direct sum  $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ , where  $\mathfrak{h}_+, \mathfrak{h}_-$  are Lie subalgebras of  $\mathfrak{h}$ . In fact, by observing that the subspaces  $\mathfrak{g}_{\pm}^{\perp}$  are  $\tau$ -*representation spaces* of  $G_{\pm}$ , we define the semidirect products

$$H_{\pm} = G_{\pm} \ltimes_{\tau} \mathfrak{g}_{\pm}^{\perp}$$

with

$$(g_{\pm}, X_{\pm}^{\perp}) \cdot (k_{\pm}, Y_{\pm}^{\perp}) := \left( g_{\pm} k_{\pm}, \tau_{k_{\pm}^{-1}} X_{\pm}^{\perp} + Y_{\pm}^{\perp} \right),$$

and the semidirect sum Lie algebras

$$\mathfrak{h}_{\pm} = \mathfrak{g}_{\pm} \ltimes_{\tau} \mathfrak{g}_{\pm}^{\perp}$$

with the Lie bracket

$$[(X_{\pm}, Y_{\pm}^{\perp}), (X'_{\pm}, Y'_{\pm}^{\perp})] = \left( [X_{\pm}, X'_{\pm}], \text{ad}_{X_{\pm}}^{\tau} Y'_{\pm}^{\perp} - \text{ad}_{X'_{\pm}}^{\tau} Y_{\pm}^{\perp} \right) \in \mathfrak{h}_{\pm}. \quad (7)$$

Then, we have the factorization

$$H = H_+ H_- ,$$

and the decomposition in direct sum of Lie subalgebras

$$\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_- .$$

In addition, the restriction of the bilinear form (4) to the subspaces  $\mathfrak{h}_+, \mathfrak{h}_-$  vanish

$$\left( (X_\pm, Y_\pm^\perp), (X'_\pm, Y'^\perp_\pm) \right)_\mathfrak{h} = (X_\pm, Y'^\perp_\pm)_\mathfrak{g} + (Y_\pm^\perp, X'_\pm)_\mathfrak{g} = 0$$

meaning that the  $\mathfrak{h}_+, \mathfrak{h}_-$  are isotropic subspaces of  $\mathfrak{h}$ . These results are summarized in the following proposition.

**Proposition:** *The Lie algebras  $\mathfrak{h}, \mathfrak{h}_+, \mathfrak{h}_-$  endowed with the bilinear form (4) compose a Manin triple  $(\mathfrak{h}, \mathfrak{h}_+, \mathfrak{h}_-)$ .*

Hence, the Lie subalgebras  $\mathfrak{h}_\pm = \mathfrak{g}_\pm \ltimes_\tau \mathfrak{g}_\pm^\perp$  are a Lie bialgebras, and the factors  $H_\pm = G_\pm \ltimes_\tau \mathfrak{g}_\pm^\perp$  are Poisson-Lie groups.

The factorization  $H = H_+ H_-$  means that each  $(g, X) \in H$  can be written as

$$(g, X) = \left( g_+, \tilde{\tau}_{g_-} \left( \Pi_{\mathfrak{g}_+^\perp} X \right) \right) \cdot \left( g_-, \Pi_{\mathfrak{g}_+^\perp} \left( Id - \tau_{g_-^{-1}} \tilde{\tau}_{g_-} \Pi_{\mathfrak{g}_+^\perp} \right) X \right) \quad (8)$$

with  $\tilde{\tau} : G_\mp \times \mathfrak{g}_\pm^\perp \rightarrow \mathfrak{g}_\pm^\perp$  defined in (2).

Let us name  $\gamma : \mathfrak{h} \longrightarrow \mathfrak{h}^*$  the linear bijection induced by the bilinear form  $(\cdot, \cdot)_\mathfrak{h}$ , then it also produces the linear bijections  $\gamma(\mathfrak{h}_\pm) = \mathfrak{h}_\mp^*$ .

### 3.3 Dressing action

Following ref. [7], we write the Lie bracket in the double Lie algebra  $(\mathfrak{h}, \mathfrak{h}_+, \mathfrak{h}_-)$  as

$$[(X_-, Y_-^\perp), (X_+, Y_+^\perp)] = (X_+, Y_+^\perp)^{(X_-, Y_-^\perp)} + (X_-, Y_-^\perp)^{(X_+, Y_+^\perp)}$$



with  $(X_+, Y_+^\perp)^{(X_-, Y_-^\perp)} \in \mathfrak{h}_+$  and  $(X_+, Y_+^\perp)^{(X_-, Y_-^\perp)} \in \mathfrak{h}_-$ . Therefore, from the Lie algebra structure (3) we get

$$\begin{cases} (X_-, Y_-^\perp)^{(X_+, Y_+^\perp)} = (X_-^{X_+}, \Pi_{\mathfrak{g}^\perp} \text{ad}_{X_-}^\tau Y_+^\perp - \Pi_{\mathfrak{g}^\perp} \text{ad}_{X_+}^\tau Y_-^\perp) \\ (X_+, Y_+^\perp)^{(X_-, Y_-^\perp)} = (X_+^{X_-}, \Pi_{\mathfrak{g}^\perp} \text{ad}_{X_-}^\tau Y_+^\perp - \Pi_{\mathfrak{g}^\perp} \text{ad}_{X_+}^\tau Y_-^\perp) \end{cases} \quad (9)$$

Let us consider the factorization for  $(g, X) \in H$  given in (8), and any couple  $(g_+, X_+^\perp) \in H_+$ ,  $(g_-, X_-^\perp) \in H_-$ . Since the product  $(g_-, X_-^\perp) \cdot (g_+, X_+^\perp)$  is also in  $H$ , it can be decomposed as above. Then, we write

$$(g_-, X_-^\perp) \cdot (g_+, X_+^\perp) = (g_+, X_+^\perp)^{(g_-, X_-^\perp)} \cdot (g_-, X_-^\perp)^{(g_+, X_+^\perp)}$$

for  $(g_+, X_+^\perp)^{(g_-, X_-^\perp)} \in H_+$  and  $(g_-, X_-^\perp)^{(g_+, X_+^\perp)} \in H_-$  given by

$$\begin{cases} (g_+, X_+^\perp)^{(g_-, X_-^\perp)} = (g_+^{g_-}, \tilde{\tau}_{g_-} \left( \Pi_{\mathfrak{g}^\perp} \tau_{g_+} X_-^\perp + X_+^\perp \right)) \\ (g_-, X_-^\perp)^{(g_+, X_+^\perp)} = (g_-^{g_+}, \Pi_{\mathfrak{g}^\perp} \tau_{g_+} X_-^\perp - \Pi_{\mathfrak{g}^\perp} \tau_{(g_-^{g_+})^{-1}} \tilde{\tau}_{g_-} \left( X_+^\perp + \Pi_{\mathfrak{g}^\perp} \tau_{g_+} X_-^\perp \right)) \end{cases}$$

The assignment  $H_- \times H_+ \longrightarrow H_+$  and  $H_+ \times H_- \longrightarrow H_-$  defined by the above relations are indeed *actions*, the so-called *dressing actions* (see [12] and [7]). In fact,  $H_- \times H_+ \longrightarrow H_+$  such that  $((g_-, X_-^\perp), (g_+, X_+^\perp)) \longmapsto (g_+, X_+^\perp)^{(g_-, X_-^\perp)}$  is a *right action* of  $H_-$  on  $H_+$  and, reciprocally,  $((g_+, X_+^\perp), (g_-, X_-^\perp)) \longmapsto (g_-, X_-^\perp)^{(g_+, X_+^\perp)}$  is a *left action* of  $H_+$  on  $H_-$ .

The infinitesimal generator of the dressing actions can be derived by considering the action of exponential elements  $(g_-, X_-^\perp) = \text{Exp} \left( t (X_-, Y_-^\perp) \right)$  and  $(g_+, X_+^\perp) = \text{Exp} \left( t (X_+, Y_+^\perp) \right)$  to get

$$\begin{cases} (g_+, X_+^\perp)^{(X_-, Y_-^\perp)} = (g_+^{X_-}, \Pi_{\mathfrak{g}^\perp} \text{ad}_{X_-}^\tau X_+^\perp + \Pi_{\mathfrak{g}^\perp} \tau_{g_+} Y_-^\perp) \\ (g_-, X_-^\perp)^{(X_+, Y_+^\perp)} = (g_-^{X_+}, \Pi_{\mathfrak{g}^\perp} \tau_{g_-} \tilde{\tau}_{g_+} \left( \Pi_{\mathfrak{g}^\perp} \text{ad}_{X_+}^\tau X_-^\perp - Y_+^\perp \right) - \Pi_{\mathfrak{g}^\perp} \text{ad}_{X_+}^\tau X_-^\perp) \end{cases} \quad (10)$$

and making the derivative at  $t = 0$ , we recover the result obtained in (9).

Also one may show that

$$\begin{cases} (X_+, X_+^\perp)^{(g_-, X_-^\perp)} = \left( X_+^{g_-}, \tilde{\tau}_{g_-} X_+^\perp - \tilde{\tau}_{g_-} \Pi_{\mathfrak{g}_+^\perp} \text{ad}_{X_+}^\tau X_-^\perp \right) \\ (X_-, X_-^\perp)^{(g_+, X_+^\perp)} = \left( X_-^{g_+}, \tilde{\tau}_{g_+} X_-^\perp + \Pi_{\mathfrak{g}_+^\perp} \text{ad}_{X_-^{g_+}}^\tau X_+^\perp \right) \end{cases}$$

which are relevant for the explicit form of the crossed adjoint actions

$$\begin{cases} \text{Ad}_{(g_+, X_+^\perp)^{-1}}^H (X_-, Y_-^\perp) = (g_+, X_+^\perp)^{-1} (g_+, X_+^\perp)^{(X_-, Y_-^\perp)} \\ \quad + (X_-, Y_-^\perp)^{(g_+, X_+^\perp)} \\ \text{Ad}_{(g_-, X_-^\perp)}^H (X_+, Y_+^\perp) = (g_-, X_-^\perp)^{(X_+, Y_+^\perp)} (g_-, X_-^\perp)^{-1} \\ \quad + (X_+, Y_+^\perp)^{(g_-, X_-^\perp)} \end{cases}$$

that are equivalent to

$$\begin{cases} \text{Ad}_{(g_+, X_+^\perp)^{-1}}^H (X_-, Y_-^\perp) = \left( \text{Ad}_{g_+}^G X_-, \text{ad}_{\text{Ad}_{g_+}^G X_-}^\tau X_+^\perp + \tau_{g_+} Y_-^\perp \right) \\ \text{Ad}_{(g_-, X_-^\perp)}^H (X_+, Y_+^\perp) = \left( \text{Ad}_{g_-}^G X_+, -\text{ad}_{\text{Ad}_{g_-}^G X_+}^\tau \tau_{g_-} X_-^\perp + \tau_{g_-} Y_+^\perp \right) \end{cases} \quad (11)$$

With this expressions we are ready to get the Poisson-Lie bivector on  $H_\pm$ .

### 3.4 Lie bialgebra and Poisson-Lie structures

Let us now work out the Lie bialgebra structures on  $\mathfrak{h}_\pm = \mathfrak{g}_\pm \oplus \mathfrak{g}_\pm^\perp$ , and the associated Poisson-Lie groups ones on  $H_\pm = G_\pm \oplus \mathfrak{g}_\pm^\perp$ . The Lie bracket on these semidirect sums were defined in (7) and, in order to define the *Lie cobracket*  $\delta : \mathfrak{h}_\pm \longrightarrow \mathfrak{h}_\pm \otimes \mathfrak{h}_\pm$ , we introduce a bilinear form on  $\mathfrak{h} \otimes \mathfrak{h}$  from the bilinear form (4) as

$$(X \otimes Y, U \otimes V)_{\mathfrak{h} \otimes \mathfrak{h}} := (X, U)_\mathfrak{h} (Y, V)_\mathfrak{h}$$

$\forall X, Y, U, V \in \mathfrak{h}$ . Thus, the Lie cobracket  $\delta$  arises from the relation

$$\begin{aligned} & ((X'_\mp, Y'^\perp_\mp) \otimes (X''_\mp, Y''^\perp_\mp), \delta(X_\pm, Y_\pm^\perp))_{\mathfrak{h} \otimes \mathfrak{h}} \\ &= ([ (X'_\mp, Y'^\perp_\mp), (X''_\mp, Y''^\perp_\mp) ], (X_\pm, Y_\pm^\perp))_\mathfrak{h} \end{aligned}$$

that in  $2 \times 2$  block matrix form means

$$\delta(X_{\pm}, Y_{\pm}^{\perp}) = \begin{pmatrix} \delta(Y_{\pm}^{\perp}) & \tau^*(X_{\pm}) \\ -\tau^*(X_{\pm}) & 0 \end{pmatrix}$$

where  $\delta : \mathfrak{g}_{\pm}^{\perp} \longrightarrow \mathfrak{g}_{\pm}^{\perp} \otimes \mathfrak{g}_{\pm}^{\perp}$  such that

$$(X'_{\mp} \otimes X''_{\mp}, \delta(Y_{\pm}^{\perp}))_{\mathfrak{g}} := ([X'_{\mp}, X''_{\mp}], Y_{\pm}^{\perp})_{\mathfrak{g}}$$

and  $\tau^* : \mathfrak{g}_{\pm} \longrightarrow \mathfrak{g}_{\pm}^{\perp} \otimes \mathfrak{g}_{\pm}$  is

$$(X'_{\mp} \otimes Y''_{\mp}^{\perp}, \tau^*(X_{\pm}))_{\mathfrak{g}} := (\text{ad}_{X'_{\mp}}^{\tau} Y''_{\mp}^{\perp}, X_{\pm})_{\mathfrak{g}}$$

### 3.4.1 The Poisson-Lie bivector on $H_+$

The Poisson-Lie bivector  $\pi_+ \in \Gamma(T^{\otimes 2}H_+)$ , the sections of the vector bundle  $T^{\otimes 2}H_+ \longrightarrow H_+$ , is defined by the relation

$$\begin{aligned} & \left\langle \gamma(X'_{-}, Y_{-}^{\perp})(g_{+}, Y_{+}^{\perp})^{-1} \otimes \gamma(X''_{-}, Y''_{-}^{\perp})(g_{+}, Y_{+}^{\perp})^{-1}, \pi_{+}(g_{+}, Z_{+}^{\perp}) \right\rangle \\ &= \left( \Pi_{-} \text{Ad}_{(g_{+}, Z_{+}^{\perp})}^H (X'_{-}, Y_{-}^{\perp}), \Pi_{+} \text{Ad}_{(g_{+}, Z_{+}^{\perp})}^H (X''_{-}, Y''_{-}^{\perp}) \right)_{\mathfrak{h}} \end{aligned} \quad (12)$$

In order to simplify the notation we introduce the projectors  $\mathbb{A}_{\pm}^G(g)$ , defined as

$$\mathbb{A}_{\pm}^G(g) := \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}_{\pm}} \text{Ad}_g^G$$

such that

$$\begin{cases} \mathbb{A}_{\pm}^G(g) \mathbb{A}_{\pm}^G(g) = \mathbb{A}_{\pm}^G(g) \\ \mathbb{A}_{\mp}^G(g) \mathbb{A}_{\pm}^G(g) = 0 \\ \mathbb{A}_{+}^G(g) + \mathbb{A}_{-}^G(g) = Id \end{cases}$$

which will be used in the following.

By using the expressions obtained in eq. (11), it takes the explicit form

$$\begin{aligned} & \left\langle \gamma(X'_{-}, Y_{-}^{\perp})(g_{+}, Z_{+}^{\perp})^{-1} \otimes \gamma(X''_{-}, Y''_{-}^{\perp})(g_{+}, Z_{+}^{\perp})^{-1}, \pi_{+}(g_{+}, Z_{+}^{\perp}) \right\rangle \\ &= \left( (X'_{-}, Y_{-}^{\perp}), \left( -\mathbb{A}_{-}^G(g_{+}^{-1}) X''_{-}, \tau_{g_{+}} \Pi_{\mathfrak{g}_{+}^{\perp}} \left( \tau_{g_{+}^{-1}} Y''_{-}^{\perp} + \text{ad}_{\Pi_{\mathfrak{g}_{-}} \text{Ad}_{g_{+}^{-1}}^G X''_{-}}^{\tau} Z_{+}^{\perp} \right) \right) \right)_{\mathfrak{h}} \end{aligned} \quad (13)$$

Introducing the linear operator  $\pi_{+(g_+, Z_+^\perp)}^R : \mathfrak{h}_- \longrightarrow \mathfrak{h}_+$  such that

$$\begin{aligned} & \left\langle \gamma(X'_-, Y'^\perp_-) (g_+, Z_+^\perp)^{-1} \otimes \gamma(X''_-, Y''^\perp_-) (g_+, Z_+^\perp)^{-1}, \pi_+(g_+, Z_+^\perp) \right\rangle \\ &= \left( (X'_-, Y'^\perp_-), \pi_{+(g_+, Z_+^\perp)}^R(X''_-, Y''^\perp_-) \right)_{\mathfrak{h}} \end{aligned}$$

we get, in components in  $\mathfrak{g}_+ \oplus \mathfrak{g}_+^\perp$ , the operator block matrix

$$\pi_{+(g_+, Z_+^\perp)}^R \begin{pmatrix} X''_- \\ Y''^\perp_- \end{pmatrix} = \begin{pmatrix} \mathbb{A}_+^G(g_+^{-1}) & 0 \\ \tau_{g_+} \Pi_+^\perp \tilde{\varphi}(Z_+^\perp) \Pi_- \text{Ad}_{g_+^{-1}}^G & \tau_{g_+} \Pi_+^\perp \tau_{g_+^{-1}} \end{pmatrix} \begin{pmatrix} X''_- \\ Y''^\perp_- \end{pmatrix}$$

Here we introduced the map  $\tilde{\varphi} : \mathfrak{g} \longrightarrow \text{End}_{vec}(\mathfrak{g})$  such that

$$\tilde{\varphi}(Z_+^\perp) X''_- := \text{ad}_{X''_-}^\tau Z_+^\perp \quad (14)$$

For a couple of functions  $\mathcal{F}, \mathcal{H}$  on  $H_+ = G_+ \ltimes \mathfrak{g}_+^\perp$  and the definition of the Poisson bracket in terms of the bivector  $\pi_H^+$

$$\{\mathcal{F}, \mathcal{H}\}_{PL}(g_+, Z_+^\perp) = \langle d\mathcal{F} \wedge d\mathcal{H}, \pi_+(g_+, Z_+^\perp) \rangle$$

we use the expression (13) to obtain

$$\begin{aligned} & \{\mathcal{F}, \mathcal{H}\}_{PL}(g_+, Z_+^\perp) \\ &= \langle g_+ \mathbf{d}\mathcal{H}, \bar{\psi}(\delta\mathcal{F}) \rangle - \langle g_+ \mathbf{d}\mathcal{F}, \bar{\psi}(\delta\mathcal{H}) \rangle + \langle \psi(Z_+^\perp), [\bar{\psi}(\delta\mathcal{F}), \bar{\psi}(\delta\mathcal{H})] \rangle \end{aligned}$$

for  $d\mathcal{F} = (\mathbf{d}\mathcal{F}, \delta\mathcal{F}) \in T^*H_+$ .

Also, from this Poisson-Lie bivector we can retrieve the infinitesimal generator of the dressing action (10) by the relation

$$(g_+, Z_+^\perp)^{(X_-, Y_-^\perp)} = \left( R_{(g_+, Z_+^\perp)} \right)_* \left( \pi_{+(g_+, Z_+^\perp)}^R(X_-, Y_-^\perp) \right) \quad (15)$$

It is worth to recall that the symplectic leaves of the Poisson-Lie structure are the orbits of the dressing actions, the integral submanifolds of the dressing vector fields, and  $(e, 0) \in H_+$  is a 1-point orbit.

### 3.4.2 The Poisson-Lie bivector on $H_-$ .

Analogously to the previous definition, the Poisson-Lie bivector  $\pi_- \in \Gamma(T^{\otimes 2}H_-)$ , is defined by the relation

$$\begin{aligned} & \left\langle (g_-, Z_-^\perp)^{-1} \gamma(X'_+, Y'^\perp_+) \otimes (g_-, Z_-^\perp)^{-1} \gamma(X''_+, Y''^\perp_+), \pi_-(g_-, Z_-^\perp) \right\rangle \\ &= \left( \Pi_- \text{Ad}_{(g_-, Z_-^\perp)}^H(X'_+, Y'^\perp_+), \Pi_+ \text{Ad}_{(g_-, Z_-^\perp)}^H(X''_+, Y''^\perp_+) \right)_{\mathfrak{h}} \end{aligned} \quad (16)$$

From the relation (11), we get

$$\begin{aligned} & \left\langle (g_-, Z_-^\perp)^{-1} \gamma(X'_+, Y'^\perp_+) \otimes (g_-, Z_-^\perp)^{-1} \gamma(X''_+, Y''^\perp_+), \pi_-(g_-, Z_-^\perp) \right\rangle \\ &= (\mathbb{A}_+^G(g_-) X'_+, Y'^\perp_+)_{\mathfrak{g}} + (Y'^\perp_+, \mathbb{A}_-^G(g_-) X''_+)_{\mathfrak{g}} - (Z_-^\perp, [X'_+, X''_+])_{\mathfrak{g}} \\ & \quad + (Z_-^\perp, [X'_+, \mathbb{A}_-^G(g_-) X''_+])_{\mathfrak{g}} + (Z_-^\perp, [\mathbb{A}_-^G(g_-) X'_+, X''_+])_{\mathfrak{g}} \end{aligned}$$

The PL bracket for functions  $\mathcal{F}, \mathcal{H}$  on  $H_- = G_- \ltimes \mathfrak{g}^\perp$  is obtained by identifying the differential  $d\mathcal{F}, d\mathcal{H}$  with  $(g_-, Z_-^\perp)^{-1} \gamma(X'_+, Y'^\perp_+)$ ,  $(g_-, Z_-^\perp)^{-1} \gamma(X''_+, Y''^\perp_+)$  respectively. Then, making explicit the left translation we obtain

$$\begin{aligned} \{\mathcal{F}, \mathcal{H}\}_{PL}(g_-, Z_-^\perp) &= -\langle g_- \mathbf{d}\mathcal{H}, \mathbb{A}_-^G(g_-) \bar{\psi}(\delta\mathcal{F}) \rangle \\ & \quad + \langle g_- \mathbf{d}\mathcal{F}, \mathbb{A}_-^G(g_-) \bar{\psi}(\delta\mathcal{H}) \rangle \\ & \quad - \langle \psi(Z_-^\perp), [\bar{\psi}(\delta\mathcal{F}), \bar{\psi}(\delta\mathcal{H})] \rangle \end{aligned}$$

Alternatively, we may write the PL bivector in terms of the associated linear operator  $\pi_{-(h_-, Z^\perp)}^L : \mathfrak{h}_-^* \longrightarrow \mathfrak{h}_-$  defined from

$$\begin{aligned} & \left\langle (g_-, Z_-^\perp)^{-1} \gamma(X'_+, Y'^\perp_+) \otimes (g_-, Z_-^\perp)^{-1} \gamma(X''_+, Y''^\perp_+), \pi_-(g_-, Z_-^\perp) \right\rangle \\ &= \left( (X'_+, Y'^\perp_+), \pi_{-(g_-, Z_-^\perp)}^L(X''_+, Y''^\perp_+) \right)_{\mathfrak{h}} \end{aligned}$$

we get, in terms of components in the direct sum, that

$$\begin{aligned} & \pi_{-(g_-, Z_-^\perp)}^L \left( \gamma \begin{pmatrix} X''_+ \\ Y''^\perp_+ \end{pmatrix} \right) \\ &= \begin{pmatrix} \Pi_- (\mathbb{A}_+^G(g_-)) & 0 \\ \tilde{\varphi}(Z_-^\perp) \mathbb{A}_+^G(g_-) - \tau_{g_-^{-1}} \Pi_+^\perp \tau_{g_-} \tilde{\varphi}(Z_-^\perp) & \tau_{g_+} \Pi_+^\perp \tau_{g_+^{-1}} \end{pmatrix} \begin{pmatrix} X''_+ \\ Y''^\perp_+ \end{pmatrix} \end{aligned}$$

where we used  $\tilde{\varphi} : \mathfrak{g} \longrightarrow \text{End}_{vec}(\mathfrak{g})$  introduced in (14), such that

$$\tilde{\varphi}(Z_-^\perp) X_+ := \text{ad}_{X_+}^\tau Z_-^\perp$$

The infinitesimal generator of the dressing action are then

$$(g_-, Z_-^\perp)^{(X_+, Y_+^\perp)} = \left( \pi_{-(e,0)}^L(\gamma(X_+, Y_+^\perp)) \right) (g_-, Z_-^\perp) \quad (17)$$

retrieving the result obtained in (10).

## 4 Integrability on $\mathfrak{g}$ from AKS on $\mathfrak{h}$

Despite AKS integrable system theory also works in absence of an Ad-invariant bilinear form [10],[11], the  $\text{Ad}^H$ -invariant bilinear form (4) on the semidirect product  $\mathfrak{h} = \mathfrak{g} \ltimes_\tau \mathfrak{g}$  brings back to the standard framework of AKS theory. We now briefly review it in the context of the semidirect product  $\mathfrak{g} \ltimes_\tau \mathfrak{g}$ .

Let us consider now  $\mathfrak{h}^*$  equipped with the Lie-Poisson structure. Then, turning the linear isomorphism  $\mathfrak{h} \xrightarrow{\gamma} \mathfrak{h}^*$  into a Poisson map, we define a Poisson structure on  $\mathfrak{h}$  such that for any couple of functions  $F, H : \mathfrak{h} \longrightarrow \mathbb{R}$ , it is

$$\{F, H\}_{\mathfrak{h}}(X) := \{F \circ \gamma^{-1}, H \circ \gamma^{-1}\}_{\mathfrak{h}^*}(\psi(X)) = (X, [\mathfrak{L}_F(X), \mathfrak{L}_H(X)])_{\mathfrak{h}}$$

where the Legendre transform  $\mathfrak{L}_F : \mathfrak{h} \longrightarrow \mathfrak{h}$  is defined as

$$(\mathfrak{L}_F(X), Y)_{\mathfrak{h}} = \left. \frac{dF(X + tY)}{dt} \right|_{t=0}$$

Let us now consider a function  $F : \mathfrak{h} \longrightarrow \mathbb{R}$ , and let  $\mathcal{F} := F \circ \iota_{K_-}$  with  $\iota_{K_-} : \mathfrak{h}_+ \longrightarrow \mathfrak{h}$ , for some  $K_- \in \mathfrak{h}_-$ , such that

$$\iota_{K_-}(X_+) = X_+ + K_-$$

The Legendre transform  $\mathfrak{L}_{\mathcal{F}} : \mathfrak{h}_+ \longrightarrow \mathfrak{h}_-$  relates with  $\mathfrak{L}_F$  as

$$\mathfrak{L}_{\mathcal{F}}(X_+) = \Pi_{\mathfrak{g}_-} \mathfrak{L}_F(\iota(X_+ + K_-)).$$

Therefore, the Lie-Poisson bracket on  $\mathfrak{h}_+$  of  $\mathcal{F}, \mathcal{H} : \mathfrak{h}_+ \longrightarrow \mathbb{R}$  coincides with that of  $F, H : \mathfrak{h} \longrightarrow \mathbb{R}$  restricted to  $\mathfrak{h}_+$

$$\begin{aligned} \{\mathcal{F}, \mathcal{H}\}_{\mathfrak{h}_+}(X_+) &= (X_+, [\mathfrak{L}_{\mathcal{F}}(X_+), \mathfrak{L}_{\mathcal{H}}(X_+)]_-)_{\mathfrak{h}} \\ &= \left\langle dF, \Pi_{\mathfrak{h}_+} \text{ad}_{\Pi_{\mathfrak{h}_-} \mathfrak{L}_F(X_+ + K_-)}^{\mathfrak{h}}(X_+ + K_-) \right\rangle \end{aligned}$$

where  $\Pi_{\mathfrak{h}_+} \text{ad}_{\Pi_{\mathfrak{h}_-} \mathfrak{L}_{\mathfrak{H}}(X_+ + K_-)}^{\mathfrak{h}}(X_+ + K_-)$  is the projection of the hamiltonian vector field of  $\mathfrak{H}$  on  $\mathfrak{h}_+$ . Then

$$\begin{aligned} \{\mathcal{F}, \mathcal{H}\}_{\mathfrak{h}_+}(X_+) &= \left. \{\mathbf{F}, \mathbf{H}\}_{\mathfrak{h}} \right|_+(X_+ + K_-) \\ &= \left( X_+, [\Pi_{\mathfrak{h}_-} \mathfrak{L}_{\mathbf{F}}(X_+ + K_-), \Pi_{\mathfrak{h}_-} \mathfrak{L}_{\mathbf{H}}(X_+ + K_-)]_- \right)_{\mathfrak{h}} \end{aligned} \quad (18)$$

Let  $\mathbf{F}$  be an  $\text{Ad}^H$ -invariant function on  $\mathfrak{h}$ , then the following relations holds:  
 $\forall h \in G$ ,

$$\text{Ad}_h^H \mathfrak{L}_{\mathbf{F}}(X) = \mathfrak{L}_{\mathbf{F}}(\text{Ad}_h^H X)$$

and  $\forall X \in \mathfrak{h}$ ,

$$\text{ad}_{\mathfrak{L}_{\mathbf{F}}(X)}^{\mathfrak{h}} X = 0 \implies \text{ad}_{\Pi_{\mathfrak{h}_-} \mathfrak{L}_{\mathbf{F}}(X)}^{\mathfrak{h}} X = -\text{ad}_{\Pi_{\mathfrak{h}_+} \mathfrak{L}_{\mathbf{F}}(X)}^{\mathfrak{h}} X$$

By using this results, it is easy to show the following statement.

**AKS Theorem:** *Let  $\gamma(K_-)$  be a character of  $\mathfrak{h}_+$ . Then, the restriction to the immersed submanifold  $\iota_{K_-} : \mathfrak{h}_+ \longrightarrow \mathfrak{h}$  of  $\text{Ad}^H$ -invariant functions on  $\mathfrak{h}$  gives rise to a nontrivial abelian Poisson algebra.*

**Remark:** *The condition  $\gamma(K_-) \in \text{char } \mathfrak{h}_+$  means,  $\forall X_+ \in \mathfrak{h}_+$ ,*

$$\text{ad}_{X_+}^* \gamma(K_-) = 0 \iff \Pi_{\mathfrak{h}_-} \text{ad}_{X_+}^{\mathfrak{h}} K_- = 0$$

*This condition in terms of components in  $\mathfrak{g} \oplus \mathfrak{g}$  implies*

$$\begin{cases} \Pi_{\mathfrak{h}_-} [\Pi_1 X_+, \Pi_1 K_-]_{\mathfrak{g}} = 0 \\ \Pi_{\mathfrak{h}_-}^{\perp} \text{ad}_{\Pi_1 X_+}^{\tau} \Pi_2 K_- = 0 \end{cases}$$

*where  $\Pi_i$ ,  $i = 1, 2$ , stand for the projectors on the first and second component in the semidirect sum  $\mathfrak{g} \ltimes \mathfrak{g}$ , respectively.*

The dynamics on these submanifolds is ruled by the hamiltonian vector fields arising from the process of restriction, as described in the following proposition.

**Proposition:** *The hamiltonian vector field associated to the Poisson bracket on  $\mathfrak{h}_+$ , eq. (18), is*

$$\begin{aligned} V_{\mathbf{H}}(X_+ + K_-) &= \Pi_{\mathfrak{h}_+} \text{ad}_{\Pi_{\mathfrak{h}_-} \mathfrak{L}_{\mathbf{H}}(X_+ + K_-)}^{\mathfrak{h}}(X_+ + K_-) \\ &= -\text{ad}_{\Pi_{\mathfrak{h}_+} \mathfrak{L}_{\mathbf{F}}(X_+ + K_-)}^{\mathfrak{h}}(X_+ + K_-) \end{aligned}$$

for  $X_+^\perp \in \mathfrak{h}_+^\perp$ .

The integral curves of this hamiltonian vector field, a *dressing vector field* indeed, are the orbits of a particular curve in  $H$ , as explained in the following proposition.

**Propositon:** *The Hamilton equation of motion on  $\iota_{K_-}(\mathfrak{h}_+) \subset \mathfrak{h}$  are*

$$\begin{cases} \dot{Z}(t) = V_H(Z(t)) \\ Z_\circ = Z(0) = X_{+\circ} + K_- \end{cases} \quad (19)$$

with  $Z(t) = X_+(t) + K_-$ , and  $H$  is an  $\text{Ad}^H$ -invariant function. It is solved by factorization: if  $h_+, h_- : \mathbb{R} \rightarrow H_\pm$  are curves on these groups defined by

$$\exp(t\mathfrak{L}_H(Z_\circ)) = h_+(t)h_-(t)$$

the solution of the above Hamilton equation is

$$Z(t) = \text{Ad}_{h_+^{-1}(t)}^H(Z_\circ)$$

**Proof:** The Hamilton equation of motion are

$$\dot{Z}(t) = -\text{ad}_{\Pi_{\mathfrak{h}_+}\mathfrak{L}_H(Z(t))}^{\mathfrak{h}} Z(t) \quad (20)$$

It is solved by

$$Z(t) = \text{Ad}_{h_+^{-1}(t)}^H(Z_\circ) \in H_+$$

with the curve  $t \mapsto h_+(t) \in H_+$  solving

$$h_+^{-1}(t)\dot{h}_+(t) = \Pi_{\mathfrak{h}_+}\mathfrak{L}_H(Z(t)) \quad (21)$$

Now, let us consider a curve  $t \mapsto h(t) = e^{t\mathfrak{L}_H(X_{+\circ}+K_-)} \in H$ , for constant  $X_{+\circ} + K_-$ , which solves the differential equation

$$\dot{h}(t)h^{-1}(t) = \mathfrak{L}_H(Z_\circ)$$



so, as  $H = H_+ H_-$  and  $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$  we have  $h(t) = h_+(t) h_-(t)$  with  $h_+(t) \in H_+$  and  $h_-(t) \in H_-$ , hence  $\dot{h} h^{-1} = \dot{h}_+ h_+^{-1} + \text{Ad}_{h_+}^H \dot{h}_- h_-^{-1}$  and, since  $\mathbf{H}$  is  $\text{Ad}^H$ -invariant, the equation of motion turns in

$$h_+^{-1} \dot{h}_+ + \dot{h}_- h_-^{-1} = \mathfrak{L}_{\mathbf{H}}(\text{Ad}_{h_+}^H(Z_{\circ}))$$

from where we conclude that

$$\begin{cases} h_+^{-1}(t) \dot{h}_+(t) = \Pi_{\mathfrak{h}_+} \mathfrak{L}_{\mathbf{H}}(Z(t)) \\ \dot{h}_-(t) h_-^{-1}(t) = \Pi_{\mathfrak{h}_-} \mathfrak{L}_{\mathbf{H}}(Z(t)) \end{cases}$$

Here we can see that the first equation coincides with (21), thus showing that the factor  $h_+(t)$  of the decomposition  $e^{t\mathfrak{L}_{\mathbf{H}}(X_+)} = h_+(t) h_-(t)$  solves the hamiltonian system (20). ■

In order to write the Hamilton equation (19) in terms of the components  $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{g}$ , we write  $Z(t) = X_+(t) + K_-$  as

$$Z(t) = (\Pi_1(X_+(t) + K_-), \Pi_2(X_+(t) + K_-))$$

Then, the evolution equations for each component are

$$\begin{cases} \Pi_1 \dot{X}_+(t) = [\Pi_1 X_+(t) + \Pi_1 K_-, \Pi_1 \Pi_{\mathfrak{h}_+} \mathfrak{L}_{\mathbf{H}}(Z(t))] \\ \Pi_2 \dot{X}_+(t) = \text{ad}_{\Pi_1 X_+(t) + \Pi_1 K_-}^{\tau} \Pi_2 \Pi_{\mathfrak{h}_+} \mathfrak{L}_{\mathbf{H}}(Z(t)) \\ \quad - \text{ad}_{\Pi_1 \Pi_{\mathfrak{h}_+} \mathfrak{L}_{\mathbf{H}}(Z(t))}^{\tau} (\Pi_2 X_+(t) + \Pi_2 K_-) \end{cases} \quad (22)$$

whose solutions are obtained from the components of the curve

$$Z(t) = \text{Ad}_{h_+^{-1}(t)}^H(Z_{\circ})$$

Explicitly they are

$$\begin{cases} \Pi_1 X_+(t) = \text{Ad}_{g_+^{-1}(t)}^G \Pi_1(Z_{\circ}) \\ \Pi_2 X_+(t) = \tau_{g_+^{-1}(t)} \Pi_2(Z_{\circ}) + \text{ad}_{\text{Ad}_{g_+^{-1}(t)}^G \Pi_1(Z_{\circ})}^{\tau} Y_+^{\perp}(t) \end{cases}$$

where  $(g_+(t), Y_+^\perp(t))$  is the  $H_+ = \mathfrak{g}_+ \oplus \mathfrak{g}_+^\perp$  factor of the exponential curve in  $H = G \ltimes \mathfrak{g}$ , namely

$$\text{Exp}^\cdot(t\mathfrak{L}_H(Z_o)) = \left( e^{t\Pi_1\mathfrak{L}_H(Z_o)}, -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} t^n \left( \text{ad}_{\Pi_1\mathfrak{L}_H(Z_o)}^\tau \right)^{n-1} \Pi_2\mathfrak{L}_H(Z_o) \right)$$

where  $\mathfrak{L}_H(Z_o) = (\Pi_1\mathfrak{L}_H(Z_o), \Pi_2\mathfrak{L}_H(Z_o)) \in \mathfrak{g} \oplus \mathfrak{g}$ .

Let us restrict to the subspace

$$\mathfrak{g}_2 := \{0\} \oplus \mathfrak{g} := \{(0, Y) / Y \in \mathfrak{g}\}$$

Then, the  $\text{Ad}^H$ -invariant function  $H$  reduces to an  $\tau^G$ -invariant function on  $\mathfrak{g}_2$  since

$$\text{Ad}_{(g,Z)}^H(X, Y) \Big|_{\mathfrak{g}_2} = (0, \tau_g Y)$$

then, if  $\iota : \mathfrak{g}_2 \longrightarrow \mathfrak{h}$  is the injection then  $\mathfrak{h} := H \circ \iota : \mathfrak{g}_2 \longrightarrow \mathbb{R}$  is the restriction of  $H$  to  $\mathfrak{g}_2$ . Then

$$\begin{aligned} (\mathfrak{L}_\mathfrak{h}(X), Y)_\mathfrak{h} &= \frac{dH \circ \iota(X + tY)}{dt} \Big|_{t=0} \\ &= (\mathfrak{L}_{H\iota}(X), (0, Y))_\mathfrak{h} = (\Pi_1\mathfrak{L}_{H\iota}(X), Y)_\mathfrak{g} \end{aligned}$$

so, we conclude that

$$\Pi_1\mathfrak{L}_{H\iota}(X) = \mathfrak{L}_\mathfrak{h}(X)$$

The restriction of the differential equations (22) to this subspace are

$$\Pi_2\dot{X}_+(t) = -\text{ad}_{\Pi_{\mathfrak{g}_+}\mathfrak{L}_\mathfrak{h}(\Pi_2X_+(t))}^\tau(\Pi_2Z_o)$$

that reproduces the equation integrable via AKS obtained in ref. [10],[11].

## 5 Examples of Lie algebras with no bi-invariant metrics

### 5.1 A three step nilpotent Lie algebra

Let us consider the three step nilpotent Lie algebra  $\mathfrak{g}$  generated by  $\{e_1, e_2, e_3, e_4\}$ , see ref. [10], defined by the nonvanishing Lie brackets

$$[e_4, e_1] = e_2 \quad , \quad [e_4, e_2] = e_3 \tag{23}$$

and the metric is determined by the nonvanishing pairings

$$(e_2, e_2)_{\mathfrak{g}} = (e_4, e_4)_{\mathfrak{g}} = 1 \quad , \quad (e_1, e_3)_{\mathfrak{g}} = -1$$

It can be decomposed in two different direct sums  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  or  $\mathfrak{g} = \mathfrak{g}_+^{\perp} \oplus \mathfrak{g}_-^{\perp}$  with

$$\begin{aligned} \mathfrak{g}_+ &= \text{span}\{e_2, e_3, e_4\} \quad , \quad \mathfrak{g}_+^{\perp} = \text{span}\{e_1\} \\ \mathfrak{g}_- &= \text{span}\{e_1\} \quad , \quad \mathfrak{g}_-^{\perp} = \text{span}\{e_2, e_3, e_4\} \end{aligned}$$

where  $\mathfrak{g}_+$  and  $\mathfrak{g}_-$  are Lie subalgebras of  $\mathfrak{g}$ .

The nonvanishing  $\tau$ -action action of  $\mathfrak{g}$  on itself are

$$\text{ad}_{e_2}^{\tau} e_1 = -e_4 \quad , \quad \text{ad}_{e_1}^{\tau} e_2 = e_4 \quad , \quad \text{ad}_{e_4}^{\tau} e_1 = e_2 \quad , \quad \text{ad}_{e_4}^{\tau} e_2 = e_3$$

and the Lie bracket in the semidirect sum Lie algebra  $\mathfrak{h} = \mathfrak{g} \ltimes \mathfrak{g}$ , see eq. (3), is defined by the following non trivial ones

$$\begin{aligned} [(e_4, 0), (e_1, 0)] &= (e_2, 0) & [(e_2, 0), (0, e_1)] &= (0, -e_4) \\ [(e_4, 0), (e_2, 0)] &= (e_3, 0) & [(e_4, 0), (0, e_2)] &= (0, e_3) \\ [(e_1, 0), (0, e_2)] &= (0, e_4) & [(e_4, 0), (0, e_1)] &= (0, -e_2) \end{aligned}$$

The Ad-invariant symmetric nondegenerate bilinear form  $(\cdot, \cdot)_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{R}$ , see eq. (4), has the following non trivial pairings

$$\begin{aligned} ((e_2, 0), (0, e_2))_{\mathfrak{h}} &= ((e_4, 0), (0, e_4))_{\mathfrak{h}} = 1 \\ ((e_1, 0), (0, e_3))_{\mathfrak{h}} &= ((e_3, 0), (0, e_1))_{\mathfrak{h}} = -1 \end{aligned}$$

giving rise to the *Manin triple*  $(\mathfrak{h}, \mathfrak{h}_+, \mathfrak{h}_-)$  with

$$\begin{aligned} \mathfrak{h}_+ &= \mathfrak{g}_+ \oplus \mathfrak{g}_+^{\perp} = \text{span}\{(e_2, 0), (e_3, 0), (e_4, 0), (0, e_3)\} \\ \mathfrak{h}_- &= \mathfrak{g}_- \oplus \mathfrak{g}_-^{\perp} = \text{span}\{(e_1, 0), (0, e_1), (0, e_2), (0, e_4)\} \end{aligned}$$

The Lie subalgebras  $\mathfrak{h}_+, \mathfrak{h}_-$  are indeed *Lie bialgebras* with the non vanishing Lie brackets and cobrackets

$$\begin{aligned} \mathfrak{h}_+ &\longrightarrow \begin{cases} [(e_4, 0), (e_2, 0)] = (e_3, 0) \\ \delta_+(e_4, 0) = (e_3, 0) \wedge (0, e_3) - (0, e_3) \wedge (e_2, 0) \end{cases} \\ \mathfrak{h}_- &\longrightarrow \begin{cases} [(e_1, 0), (0, e_2)] = (0, e_4) \\ \delta_-(0, e_1) = (0, e_2) \wedge (0, e_4) \end{cases} \end{aligned} \tag{24}$$

so the associated Lie groups  $H_{\pm} = G_{\pm} \oplus \mathfrak{g}_{\pm}^{\perp}$  become into *Poisson Lie groups*. We determine the Poisson Lie structure in the matrix representation of this Lie algebra (23) in a four dimensional vector space where the Lie algebra generators, in terms of the  $4 \times 4$  elementary matrices  $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ , are

$$e_1 = E_{34} \quad , \quad e_2 = E_{24} \quad , \quad e_3 = E_{14} \quad , \quad e_4 = E_{12} + E_{23}$$

In this representation, any vector  $X = (x_1, x_2, x_3, x_4)$  of this Lie algebra is 4-step nilpotent,  $X^4 = 0$ .

The *Poisson-Lie bivector*  $\pi_+$  on  $H_+$  is defined by the relation (12). Since the exponential map is surjective on nilpotent Lie groups, we may write

$$\begin{aligned} (g_+, Z_+^{\perp}) &= (e^{u_2 e_2 + u_3 e_3 + u_4 e_4}, z_3 e_3) \\ (X_-, Y_-^{\perp}) &= (x_1 e_1, y_1 e_1 + y_2 e_2 + y_4 e_4) \end{aligned}$$

to get

$$\begin{aligned} &\left\langle \gamma(X'_-, Y_-^{\perp}) (g_+, Z_+^{\perp})^{-1} \otimes \gamma(X''_-, Y_-^{\perp}) (g_+, Z_+^{\perp})^{-1}, \pi_+(g_+, Z_+^{\perp}) \right\rangle \\ &= \frac{1}{2} y'_1 u_4^2 x''_1 - \frac{1}{2} x'_1 u_4^2 y''_1 + x'_1 u_4 y''_2 - y'_2 u_4 x''_1 \end{aligned}$$

Introducing  $\pi_+^R : G \longrightarrow \mathfrak{h}_+ \otimes \mathfrak{h}_+$  as  $\pi_+^R(g_+, Z_+^{\perp}) = \left( R_{(g_+, Z_+^{\perp})}^{-1} \right)^{\otimes 2}_* \pi_+(g_+, Z_+^{\perp})$ , we get

$$\begin{aligned} \pi_+^R(g_+, Z_+^{\perp}) &= \frac{1}{2} u_4^2 (e_3, 0) \otimes (0, e_3) - \frac{1}{2} u_4^2 (0, e_3) \otimes (e_3, 0) \\ &\quad - u_4 (0, e_3) \otimes (e_2, 0) + u_4 (e_2, 0) \otimes (0, e_3) \end{aligned}$$

The dressing vector can be obtained from the PL bivector by using eq. (15) to get

$$(g_+, Z_+^{\perp})^{(X''_-, Y_-^{\perp})} = \left( -u_4 x''_1 e_2 - \frac{1}{2} u_4^2 x''_1 e_3, \left( \frac{1}{2} u_4^2 y''_1 - u_4 y''_2 \right) e_3 \right)$$

and, the cobracket  $\delta_+ : \mathfrak{h}_+ \longrightarrow \mathfrak{h}_+ \wedge \mathfrak{h}_+$  defined from  $(\pi_+^R)_{*(e,0)}$  reproducing (24). Since  $\mathfrak{h}_+$  has only one non trivial Lie bracket, see eq. (24), there is no an object  $r \in \mathfrak{h}_+ \otimes \mathfrak{h}_+$  making  $\delta_+$  in a coboundary.

In order to determine *the Poisson-Lie structure on  $H_-$* , we write the elements in  $H_-$  and  $\mathfrak{h}_+$  as

$$\begin{aligned}(g_-, Z_-^\perp) &= (e^{u_1 e_1}, z_1 e_1 + z_2 e_2 + z_4 e_4) \\ (X_+, Y_+^\perp) &= (x_2 e_2 + x_3 e_3 + x_4 e_4, y_3 e_3)\end{aligned}$$

and PL bivector  $\pi_-$  on  $H_-$  defined by the relation (16) is

$$\left\langle (g_-, Z_-^\perp)^{-1} (\gamma(X'_+, Y'^+_\perp) \otimes \gamma(X''_+, Y''^+_\perp)), \pi_- (g_-, Z_-^\perp) \right\rangle = x'_4 z_1 x''_2 - x'_2 z_1 x''_4$$

Writing it in terms of the dressing vector as in eq. (17) we obtain

$$(g_-, Z_-^\perp)^{(X''_+, Y''^+_\perp)} = (0, -z_1 x''_4 e_2 + z_1 x''_2 e_4)$$

Introducing  $\pi_-^L : G \longrightarrow \mathfrak{h}_- \otimes \mathfrak{h}_-$  as  $\pi_-^L (g_-, Z_-^\perp) = \left( L_{(g_-, Z_-^\perp)^{-1}} \right)_*^{\otimes 2} \pi_- (g_-, Z_-^\perp)$ , we have

$$\pi_-^L (g_-, Z_-^\perp) = z_1 (0, e_4) \otimes (0, e_2) - z_1 (0, e_2) \otimes (0, e_4)$$

The cobracket on  $\mathfrak{h}_-$ ,  $\delta_- : \mathfrak{h}_- \longrightarrow \mathfrak{h}_- \wedge \mathfrak{h}_-$  given in (24) can be easily retrieved as  $\delta_- := -(\pi_-^L)_{*(e,0)}$ . Again, the only nonvanishing Lie bracket in  $\mathfrak{h}_-$  does not allows for some object  $r$  in  $\mathfrak{h}_- \otimes \mathfrak{h}_-$  giving rise to that cobracket.

The decomposition  $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$  implies the factorization of the Lie group  $H$  as  $H_+ H_-$  such that

$$\begin{aligned}& (e^{u_1 e_1 + u_2 e_2 + u_3 e_3 + u_4 e_4}, z_1 e_1 + z_2 e_2 + z_3 e_3 + z_4 e_4) \\ &= \left( e^{(u_2 - \frac{1}{2} u_1 u_4) e_2 + (u_3 - \frac{1}{12} u_1 u_4^2) e_3 + u_4 e_4}, z_3 e_3 \right) \cdot (e^{u_1 e_1}, z_1 e_1 + z_2 e_2 + z_4 e_4)\end{aligned}$$

and from here we get the reciprocal dressing actions

$$\left\{ \begin{aligned} & (e^{u_2 e_2 + u_3 e_3 + u_4 e_4}, z_3 e_3)^{(e^{u_1 e_1}, z_1 e_1 + z_2 e_2 + z_4 e_4)} \\ &= \left( e^{(u_2 - u_1 u_4) e_2 + (u_3 - \frac{1}{2} u_1 u_4^2) e_3 + u_4 e_4}, \left( \frac{1}{2} z_1 u_4^2 - z_2 u_4 + z_3 \right) e_3 \right) \\ & (e^{u_1 e_1}, z_1 e_1 + z_2 e_2 + z_4 e_4)^{(e^{u_2 e_2 + u_3 e_3 + u_4 e_4}, z_3 e_3)} \\ &= (e^{u_1 e_1}, z_1 e_1 + (z_2 - z_1 u_4) e_2 + (z_1 u_2 + z_4) e_4) \end{aligned} \right.$$

### 5.1.1 AKS Integrable system

In order to produce an AKS integrable system, we consider an  $\text{ad}^{\mathfrak{h}}$ -invariant function  $H$  on  $\mathfrak{h}$ . We consider  $(\cdot, \cdot)_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{R}$

$$H(X, Y) = \frac{1}{2} \left( ((X, Y), (X, Y))_{\mathfrak{h}} \right) = (X, Y)_{\mathfrak{g}}$$

so

$$\mathfrak{L}_H(X, Y) = (X, Y)$$

The hamiltonian vector field is

$$V_H(X, Y) = [(X_-, Y_-^\perp), (X_+, Y_+^\perp)]$$

The condition  $\psi(X_-, Y_-^\perp) \in \text{char}(\mathfrak{h}_+)$ , is satisfied provided the component in  $(0, e_1)$  vanish, so the allowed motion is performed on points

$$(X, Y) = (x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4, y_2 e_2 + y_3 e_3 + y_4 e_4)$$

and the Hamiltonian vector fields reduces to

$$V_H(Z) = -x_4 (x_1 e_2, y_2 e_3)$$

giving rise to non trivial Hamilton equation of motion  $\dot{Z}(t) = V_H(Z(t))$

$$\begin{cases} \dot{x}_2 = -x_4 x_1 \\ \dot{y}_3 = -x_4 y_2 \end{cases} \quad (25)$$

The coordinates  $x_1, x_3, x_4, y_2, y_4$  remains constants of motion, unveiling a quite simple dynamical system, namely a uniform linear motion in the plane  $(x_2, y_3)$ . Despite this simplicity, it is interesting to see how the  $H_+$  factor of the semidirect product exponential curve succeeds in yielding this linear trajectories as the orbits by the adjoint action.

In order to apply the AKS Theorem, we obtain

$$\text{Exp}^*(t \mathfrak{L}_H(X_{+0} + K_-)) = h_+(t) h_-(t)$$

with

$$\begin{aligned} h_+(t) &= \left( e^{(tx_2 - \frac{1}{2}t^2x_1x_4)e_2 + (tx_3 - \frac{1}{12}t^3x_1x_4^2)e_3 + tx_4e_4}, -\left(\frac{1}{2}t^2x_4y_2 - ty_3\right)e_3 \right) \\ h_-(t) &= \left( e^{tx_1e_1}, ty_2e_2 - \left(\frac{1}{2}t^2x_1y_2 - ty_4\right)e_4 \right) \end{aligned}$$

Thus, for the initial condition

$$Z(t_0) = (x_{10}e_1 + x_{20}e_2 + x_{30}e_3 + x_{40}e_4, y_{20}e_2 + y_{30}e_3 + y_{40}e_4)$$

the system (25) has the solution

$$Z(t) = \text{Ad}_{h_+^{-1}(t)}^H Z(t_0)$$

described by the curve

$$h_+^{-1}(t) = \left( e^{-(tx_{20} - \frac{t^2}{2}x_{10}x_{40})e_2 - (tx_{30} - \frac{t^3}{12}x_{10}x_{40}^2)e_3 - tx_{40}e_4}, \left(\frac{t^2}{2}x_{40}y_{20} - ty_{30}\right)e_3 \right)$$

The explicit form of the adjoint curve is

$$\begin{aligned} Z(t) &= (x_{10}e_1 + (x_{20} - tx_{10}x_{40})e_2 + x_{30}e_3 + x_{40}e_4, \\ &\quad y_{20}e_2 + (y_{30} - tx_{40}y_{20})e_3 + y_{40}e_4) \end{aligned}$$

solving the system of Hamilton equations (25).

## 5.2 The Lie Group $G = A_{6.34}$

This example was taken from reference [4]. Its Lie algebra  $\mathfrak{g} = \mathfrak{a}_{6.34}$  is generated by the basis  $\{e_1, e_2, e_3, e_4, e_5, e_6\}$  with the non vanishing Lie brackets

$$[e_2, e_3] = e_1 \quad , \quad [e_2, e_6] = e_3 \quad , \quad [e_3, e_6] = -e_2 \quad , \quad [e_4, e_6] = e_5$$

The Lie algebra can be represented by  $6 \times 6$  matrices, and by using the elementary matrices  $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$  they can be written as

$$\begin{cases} e_1 = -2E_{36} & , & e_2 = E_{35} & , & e_3 = E_{34} - E_{56} \\ e_4 = E_{12} & , & e_5 = E_{16} & , & e_6 = E_{26} + E_{54} - E_{45} \end{cases} ,$$

and a typical element  $g(z, x, y, p, q, \theta)$  of the associated Lie group  $G = A_{6,34}$  is

$$g(z, x, y, p, q, \theta) = \begin{pmatrix} 1 & p & 0 & 0 & 0 & q \\ 0 & 1 & 0 & 0 & 0 & \theta \\ 0 & 0 & 1 & x \sin \theta + y \cos \theta & x \cos \theta - y \sin \theta & z \\ 0 & 0 & 0 & \cos \theta & -\sin \theta & x \\ 0 & 0 & 0 & \sin \theta & \cos \theta & -y \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The group  $G$  can be factorize as  $G = G_+ G_-$  with  $G_+, G_-$  being the Lie subgroups

$$G_+ = \{g_+(x, y, z) = g(z, x, y, 0, 0, 0) / (x, y, z) \in \mathbb{R}^3\}$$

$$G_- = \{g_-(p, q, \theta) = g(0, 0, 0, p, q, \theta) / (p, q, \theta) \in \mathbb{R}^3\}$$

in such a way that, for  $g(z, x, y, p, q, \theta) \in G$ ,

$$g(z, x, y, p, q, \theta) = g_+(x, y, z) g_-(p, q, \theta) \quad .$$

By using this result we determine the dressing actions

$$\begin{cases} (g_+(x, y, z))^{g_-(p, q, \theta)} = g_+(x \cos \theta + y \sin \theta, y \cos \theta - x \sin \theta, z) \\ (g_-(p, q, \theta))^{g_+(x, y, z)} = g_-(p, q, \theta) \end{cases} \quad .$$

The Lie algebra  $\mathfrak{g}$  decompose then as  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$  with

$$\mathfrak{g}_+ = Lspan\{e_1, e_2, e_3\} \quad , \quad \mathfrak{g}_- = Lspan\{e_4, e_5, e_6\}$$

being Lie subalgebras of  $\mathfrak{g}$ . The exponential map is surjective and each element  $g(z, x, y, p, q, \theta) \in A_{6,34}$  can be written as

$$\begin{aligned} & g(z, x, y, p, q, \theta) \\ = & e^{\left(-\frac{1}{2}ze_1 + \frac{\theta}{2} \frac{(x \sin \theta - y(1 - \cos \theta))}{(1 - \cos \theta)} e_2 + \frac{\theta}{2} \frac{(x(1 - \cos \theta) + y \sin \theta)}{(1 - \cos \theta)} e_3 + pe_4 + \left(q - \frac{1}{2}p\theta\right)e_5 + \theta e_6\right)} \end{aligned}$$



Let us now introduce a bilinear form on  $\mathfrak{g}$  inherited from the non-invariant metric on  $G$  given in reference [4],

$$g = dp^2 + \left( dq - \frac{1}{2} (pd\theta + \theta dp) \right)^2 + dx^2 + dy^2 - ydx d\theta + xdy d\theta + dzd\theta$$

By taking this metric at the identity element of  $G$  we get the nondegenerate symmetric bilinear form  $(\cdot, \cdot)_{\mathfrak{g}} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$  defined as

$$\begin{aligned} & (X(x_1, x_2, x_3, x_4, x_5, x_6), X(x'_1, x'_2, x'_3, x'_4, x'_5, x'_6))_{\mathfrak{g}} \\ &= x_2 x'_2 + x_3 x'_3 + x_4 x'_4 + x_5 x'_5 + x_1 x'_6 + x_6 x'_1 \end{aligned}$$

It gives rise to the decomposition of  $\mathfrak{g}$  as  $\mathfrak{g}_+^{\perp} \oplus \mathfrak{g}_-^{\perp}$  where

$$\mathfrak{g}_+^{\perp} = Lspan\{e_1, e_4, e_5\} \quad , \quad \mathfrak{g}_-^{\perp} = Lspan\{e_2, e_3, e_6\}$$

This bilinear form defines associated  $\tau$ -action of  $\mathfrak{g}$  on itself

$$\begin{aligned} \text{ad}_{e_3}^{\tau} e_2 &= e_1 \quad , \quad \text{ad}_{e_2}^{\tau} e_3 = -e_1 \quad , \quad \text{ad}_{e_4}^{\tau} e_5 = -e_1 \quad , \quad \text{ad}_{e_2}^{\tau} e_6 = -e_3 \\ \text{ad}_{e_6}^{\tau} e_2 &= -e_3 \quad , \quad \text{ad}_{e_6}^{\tau} e_3 = e_2 \quad , \quad \text{ad}_{e_6}^{\tau} e_5 = e_4 \quad , \quad \text{ad}_{e_3}^{\tau} e_6 = e_2 \end{aligned}$$

So, let us now consider the semidirect sum  $\mathfrak{h} = \mathfrak{g} \ltimes \mathfrak{g}$  where the left  $\mathfrak{g}$  acts on the other one by the  $\tau$ -action. We have then the decomposition  $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$  with

$$\mathfrak{h}_+ = \mathfrak{g}_+ \oplus \mathfrak{g}_+^{\perp} = Lspan\{(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_4), (0, e_5)\}$$

$$\mathfrak{h}_- = \mathfrak{g}_- \oplus \mathfrak{g}_-^{\perp} = Lspan\{(e_4, 0), (e_5, 0), (e_6, 0), (0, e_2), (0, e_3), (0, e_6)\}$$

The Lie algebra structure on  $\mathfrak{h}$  is given by (3), with the fundamental Lie brackets

$$\begin{aligned} [(e_2, 0), (e_3, 0)] &= (e_1, 0) & [(e_3, 0), (0, e_2)] &= (0, e_1) \\ [(e_2, 0), (e_6, 0)] &= (e_3, 0) & [(e_3, 0), (0, e_6)] &= (0, e_2) \\ [(e_3, 0), (e_6, 0)] &= -(e_2, 0) & [(e_4, 0), (0, e_5)] &= -(0, e_1) \\ [(e_4, 0), (e_6, 0)] &= (e_5, 0) & [(e_6, 0), (0, e_2)] &= -(0, e_3) \\ [(e_2, 0), (0, e_3)] &= -(0, e_1) & [(e_6, 0), (0, e_3)] &= (0, e_2) \\ [(e_2, 0), (0, e_6)] &= -(0, e_3) & [(e_6, 0), (0, e_5)] &= (0, e_4) \end{aligned}$$

The semidirect product structure of  $H = G \ltimes \mathfrak{g}$  is defined by eq. (5).

Every element  $\left(g(z, x, y, p, q, \theta), \sum_{i=1}^6 x_i e_i\right) \in H$  can be factorized as

$$\begin{aligned} & \left(g(z, x, y, p, q, \theta), \sum_{i=1}^6 x_i e_i\right) \\ &= (g_+(z, x, y), (x_1 - x_5 p) e_1 + (x_4 + x_5 \theta) e_4 + x_5 e_5) \\ & \quad \cdot (g_-(p, q, \theta), x_2 e_2 + x_3 e_3 + x_6 e_6) \end{aligned}$$

from where we get the reciprocal *dressing actions*

$$\begin{aligned} & (g_+(x, y, z), x_1 e_1 + x_4 e_4 + x_5 e_5)^{(g_-(p, q, \theta), x_2 e_2 + x_3 e_3 + x_6 e_6)} \\ &= (g_+(x \cos \theta + y \sin \theta, y \cos \theta - x \sin \theta, z), \\ & \quad \left(\frac{1}{2} x_6 (x^2 + y^2) - (y x_2 - x x_3) + x_1 - x_5 p\right) e_1 + (x_4 + x_5 \theta) e_4 + x_5 e_5) \end{aligned}$$

and

$$\begin{aligned} & (g_-(p, q, \theta), x_2 e_2 + x_3 e_3 + x_6 e_6)^{(g_+(x, y, z), x_1 e_1 + x_4 e_4 + x_5 e_5)} \\ &= (g_-(p, q, \theta), (x_2 - x_6 y) e_2 + (x_3 + x_6 x) e_3 + x_6 e_6) \end{aligned}$$

The infinitesimal generator of these dressing action are

$$\begin{aligned} & (g_+(x, y, z), x_1 e_1 + x_4 e_4 + x_5 e_5)^{(z_4 e_4 + z_5 e_5 + z_6 e_6, z'_2 e_2 + z'_3 e_3 + z'_6 e_6)} \\ &= (y z_6 e_2 - x z_6 e_3, \left(\frac{1}{2} (x^2 + y^2) z'_6 - x_5 z_4 + x z'_3 - y z'_2\right) e_1 + x_5 z_6 e_4) \end{aligned}$$

and

$$\begin{aligned} & (g_-(p, q, \theta), x_2 e_2 + x_3 e_3 + x_6 e_6)^{(z_1 e_1 + z_2 e_2 + z_3 e_3, z'_1 e_1 + z'_4 e_4 + z'_5 e_5)} \\ &= (0, -x_6 z_3 e_2 + x_6 z_2 e_3) \end{aligned}$$

The Poisson-Lie bivector is determined from the relation (12), that for

$$\begin{aligned} (g_+, X_+^\perp) &= (g(z, x, y), x_1 e_1 + x_4 e_4 + x_5 e_5) \\ (Y_-, Y_-^\perp) &= (y_4 e_4 + y_5 e_5 + y_6 e_6, y'_2 e_2 + y'_3 e_3 + y'_6 e_6) \\ (Z_-, Z_-^\perp) &= (z_4 e_4 + z_5 e_5 + z_6 e_6, z'_2 e_2 + z'_3 e_3 + z'_6 e_6) \end{aligned}$$

we get

$$\begin{aligned} & \left\langle \gamma(Y_-, Y_-^\perp) (g_+, Z_+^\perp)^{-1} \otimes \gamma(Z_-, Z_-^\perp) (g_+, X_+^\perp)^{-1}, \pi_+(g_+, X_+^\perp) \right\rangle \\ &= y_4 x_5 z_6 + y_6 x z'_3 - y_6 x_5 z_4 - y_6 y z'_2 \\ & \quad + \frac{1}{2} y_6 (x^2 + y^2) z'_6 + y'_2 y z_6 - y'_3 x z_6 - \frac{1}{2} y'_6 (x^2 + y^2) z_6 \end{aligned}$$

Introducing the linear map  $\pi_+^R : \mathfrak{h}_- \longrightarrow \mathfrak{h}_+$  defined as

$$\begin{aligned} & \left\langle \gamma(Y_-, Y_-^\perp) (g_+, Z_+^\perp)^{-1} \otimes \gamma(Z_-, Z_-^\perp) (g_+, X_+^\perp)^{-1}, \pi_+(g_+, X_+^\perp) \right\rangle \\ &= ((Y_-, Y_-^\perp), \pi_+^R(Z_-, Z_-^\perp))_{\mathfrak{h}} \end{aligned}$$

we get that it is characterized by the matrix

$$\begin{aligned} & \pi_+^R(g(z, x, y), x_1 e_1 + x_4 e_4 + x_5 e_5) \\ &= \begin{pmatrix} 0 & 0 & x_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -x_5 & 0 & 0 & -y & x & \frac{1}{2}(x^2 + y^2) \\ 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}(x^2 + y^2) & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The Poisson-Lie bivector  $H_-$  is determined from the relation (16) for

$$\begin{aligned} (g_-, X_-^\perp) &= (g(p, q, \theta), x_2 e_2 + x_3 e_3 + x_6 e_6) \\ (Y_+, Y_+^\perp) &= (y_1 e_1 + y_2 e_2 + y_3 e_3, y'_1 e_1 + y'_4 e_4 + y'_5 e_5) \\ (Z_+, Z_+^\perp) &= (z_1 e_1 + z_2 e_2 + z_3 e_3, z'_1 e_1 + z'_4 e_4 + z'_5 e_5) \end{aligned}$$

and after some lengthy computations we get

$$\left\langle (g_-, X_-^\perp)^{-1} (\gamma(Y_+, Y_+^\perp) \otimes \gamma(Z_+, Z_+^\perp)), \pi_-(g_-, Z_-^\perp) \right\rangle = y_3 x_6 z_2 - y_2 x_6 z_3$$

It has associated a bilinear form on  $\mathfrak{h}_+$ , that in the given basis is characterized by the matrix

$$\pi_-^L(g(p, q, \theta), x_2 e_2 + x_3 e_3 + x_6 e_6) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_6 & 0 & 0 & 0 \\ 0 & x_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

### 5.2.1 AKS-Integrable system on $G = A_{6.34}$

Let us study a dynamical system on  $\mathfrak{h}$  ruled by the  $\text{ad}^{\mathfrak{h}}$ -invariant Hamilton function  $H$  on  $\mathfrak{h}$

$$H(X, X') = \frac{1}{2} \left( ((X, X'), (X, X'))_{\mathfrak{h}} \right) = (X, X')_{\mathfrak{g}}$$

Writing  $(X, X') \in \mathfrak{h}$  as

$$(X, X') = \left( \sum_{i=1}^6 x_i e_i, \sum_{i=1}^6 x'_i e_i \right)$$

it becomes in

$$H(X, X') = \frac{1}{n} (x'_2 x_2 + x'_3 x_3 + x'_4 x_4 + x'_5 x_5 + x_1 x'_6 + x_6 x'_1)$$

for which

$$\mathfrak{L}_H(X, X') = (X, X')$$

The hamiltonian vector field is then

$$V_H(X, X') = -\text{ad}^{\mathfrak{h}}_{\Pi_+ \mathfrak{L}_H(X, X')}(X, X') = [\Pi_- (X, X'), \Pi_+ (X, X')]$$

Then, the explicit computation of the Lie bracket and applying the condition  $\gamma(K_-) \in \text{char}(\mathfrak{h}_+)$ , which means that  $x'_6 = 0$ , we get the Hamiltonian vector fields

$$V_H(X, X') = (x_3 x_6 e_2 - x_2 x_6 e_3, (x_2 x'_3 - x_3 x'_2 - x_4 x'_5) e_1 + x'_5 x_6 e_4)$$

and the nontrivial equation of motions are then

$$\begin{cases} \dot{x}_2 = x_3 x_6 \\ \dot{x}_3 = -x_2 x_6 \\ \dot{x}'_1 = (x_2 x'_3 - x_3 x'_2 - x_4 x'_5) \\ \dot{x}'_4 = x'_5 x_6 \end{cases} \quad (26)$$

with  $x_1, x_4, x_5, x_6, x'_2, x'_3, x'_5, x'_6$  being time independent.

Let us now to integrate these equations by the AKS method. In doing so, we need to factorize the exponential curve (6)

$$\text{Exp } t \mathfrak{L}_H(X_0, X'_0) = \text{Exp } t \left( \sum_{i=1}^6 x_{i0} e_i, \sum_{i=1}^5 x'_{i0} e_i \right)$$

for some initial  $(X_0, X'_0) \in \mathfrak{h}$ .

Factorizing the exponential

$$\text{Exp } t\mathfrak{L}_H(X_0, X'_0) = h_+(t) h_-(t)$$

we obtain the curve

$$t \longmapsto h_+(t) = (g_+(t), W_+^\perp(t))$$

where

$$g_+(t) = g_+ \left( -2tx_{10}, \frac{(x_{30}(1-\cos tx_{60})+x_{20} \sin tx_{60})}{x_{60}}, \frac{(x_{20}(\cos tx_{60}-1)+x_{30} \sin tx_{60})}{x_{60}} \right)$$

and

$$W_+^\perp(t) = (tx'_{10} - 2t^2 x_{40} x'_{50}) e_1 + (tx'_4 + 2t^2 x_{60} x'_{50}) e_4 + tx'_5 e_5$$

which through the adjoint action

$$Z(t) = \text{Ad}_{h_+^{-1}(t)}^H(X_0, X'_0) = \left( \text{Ad}_{g_+^{-1}(t)}^G X_0, \tau_{g_+^{-1}(t)}(X'_0 - \text{ad}_{X_0}^\tau W_+^\perp(t)) \right)$$

gives rise to the solution of the Hamilton equations (26). In fact, after some calculations we get the nontrivial solutions

$$\begin{cases} x_2(t) = x_{20} \cos tx_{60} + x_{30} \sin tx_{60} \\ x_3(t) = x_{30} \cos tx_{60} - x_{20} \sin tx_{60} \\ x'_1(t) = (x'_{10} - tx_{40} x'_{50}) - \frac{(x'_{20} x_2(t) + x'_{30} x_3(t))}{x_{60}} + \frac{(x_{30} x'_{30} + x_{20} x'_{20})}{x_{60}} \\ x'_4(t) = x'_{40} + tx_{60} x'_{50} \end{cases}$$

### 5.3 $\mathfrak{sl}_2(\mathbb{C})$ equipped with an inner product

Let us consider the Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$ , with the associated decomposition

$$\mathfrak{sl}_2(\mathbb{C})^{\mathbb{R}} = \mathfrak{su}_2 \oplus \mathfrak{b} \tag{27}$$

where  $\mathfrak{b}$  is the subalgebra of upper triangular matrices with real diagonal and null trace, and  $\mathfrak{su}_2$  is the real subalgebra of  $\mathfrak{sl}_2(\mathbb{C})$  of antihermitean matrices.

For  $\mathfrak{su}_2$  we take the basis

$$X_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and in  $\mathfrak{b}$  this one

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad iE = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \quad , \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Killing form in  $\mathfrak{sl}_2(\mathbb{C})$  is

$$\kappa(X, Y) = \text{tr}(ad(X)ad(Y)) = 4\text{tr}(XY),$$

and the bilinear form on  $\mathfrak{sl}_2(\mathbb{C})$

$$k_0(X, Y) = -\frac{1}{4}\text{Im}\kappa(X, Y)$$

is nondegenerate, symmetric and Ad-invariant, turning  $\mathfrak{b}$  and  $\mathfrak{su}_2$  into isotropic subspaces. However, it fails in to be an *inner product* and, in consequence, it does not give rise to a Riemannian metric on the Lie group  $SL(2, \mathbb{C})$ .

By introducing the idempotent linear operator  $\mathcal{E} : \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{sl}_2(\mathbb{C})$  defined as

$$\begin{aligned} \mathcal{E}X_1 &= -E \quad , \quad \mathcal{E}X_2 = iE \quad , \quad \mathcal{E}X_3 = -H \\ \mathcal{E}E &= -X_1 \quad , \quad \mathcal{E}iE = X_2 \quad , \quad \mathcal{E}H = -X_3 \end{aligned}$$

we define the *non Ad-invariant inner product*  $g : \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathbb{R}$  such that for  $V, W \in \mathfrak{sl}_2(\mathbb{C})$  we have

$$g(V, W) := k_0(V, \mathcal{E}W) .$$

So, by left or right translations, it induces a non bi-invariant Riemannian metric on  $SL(2, \mathbb{C})$ , regarded as a real manifold.

Then, beside Iwasawa decomposition (27), we have the vector subspace direct sum decomposition  $\mathfrak{sl}_2(\mathbb{C})^{\mathbb{R}} = (\mathfrak{su}_2)^{\perp} \oplus \mathfrak{b}^{\perp}$ , where  $^{\perp}$  stands for the orthogonal subspaces referred to the inner product  $g$ . Since the basis  $\{X_1, X_2, X_3, E, (iE), H\}$  is an orthogonal one in relation with  $g$ , we have that  $(\mathfrak{su}_2)^{\perp} = \mathfrak{b}$  and  $\mathfrak{b}^{\perp} = \mathfrak{su}_2$ .

We now built an example of the construction developed in previous sections by considering  $\mathfrak{sl}_2(\mathbb{C})$  equipped with the inner product  $g : \mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathbb{R}$ . The  $\tau$ -action of  $SL(2, \mathbb{C})$  on  $\mathfrak{sl}_2(\mathbb{C})$  and of  $\mathfrak{sl}_2(\mathbb{C})$  on itself are then

$$\tau_g = \mathcal{E} \circ \text{Ad}_g \circ \mathcal{E} \quad , \quad \text{ad}_X^{\tau} = \mathcal{E} \circ \text{ad}_X \circ \mathcal{E}$$

In the remaining of this example we often denote  $\mathfrak{g}_+ = \mathfrak{su}_2$ ,  $\mathfrak{g}_- = \mathfrak{b}$ ,  $G_+ = SU(2)$  and  $G_- = B$ . Let us denote the elements  $g \in SL(2, \mathbb{C})$ ,  $g_+ \in SU(2)$  and  $g_- \in B$  as

$$g = \begin{pmatrix} \mu & \nu \\ \rho & \sigma \end{pmatrix}, \quad g_+ = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad g_- = \begin{pmatrix} a & z \\ 0 & a^{-1} \end{pmatrix},$$

with  $\mu, \sigma, \nu, \rho, \alpha, \beta, z \in \mathbb{C}$ ,  $a \in \mathbb{R}_{>0}$ , and  $\mu\sigma - \nu\rho = |\alpha|^2 + |\beta|^2 = 1$ .

The factorization  $SL(2, \mathbb{C}) = SU(2) \times B$  means that  $g = g_+ g_-$  with

$$g_+ = \frac{1}{\sqrt{|\mu|^2 + |\rho|^2}} \begin{pmatrix} \mu & -\bar{\rho} \\ \rho & \bar{\mu} \end{pmatrix}, \quad g_- = \frac{1}{\sqrt{|\mu|^2 + |\rho|^2}} \begin{pmatrix} |\mu|^2 + |\rho|^2 & \bar{\mu}\nu + \bar{\rho}\sigma \\ 0 & 1 \end{pmatrix}$$

from where we determine the dressing actions

$$g_+^{g_-} = \frac{1}{\Delta(g_+, g_-)} \begin{pmatrix} a \left( \alpha - \frac{1}{a} z \bar{\beta} \right) & \frac{1}{a} \beta \\ -\frac{1}{a} \bar{\beta} & a \left( \bar{\alpha} - \frac{1}{a} \bar{z} \beta \right) \end{pmatrix}$$

$$g_-^{g_+} = \begin{pmatrix} \Delta(g_+, g_-) & \frac{\left( a^2 \beta \bar{\alpha} + a z \bar{\alpha}^2 - a \bar{z} \beta^2 - \left( |z|^2 + \frac{1}{a^2} \right) \bar{\alpha} \beta \right)}{\Delta(g_+, g_-)} \\ 0 & \frac{1}{\Delta(g_+, g_-)} \end{pmatrix}$$

with

$$\Delta(g_+, g_-) = \sqrt{a^2 |\alpha|^2 - a (\bar{z} \alpha \beta + z \bar{\beta} \bar{\alpha}) + \left( |z|^2 + \frac{1}{a^2} \right) |\beta|^2}.$$

The dressing vector fields associated with  $X_+ = x_1 X_1 + x_2 X_2 + x_3 X_3 \in \mathfrak{g}_+$  is

$$g_-^{X_+} = \frac{\sqrt{\det X_+}}{a} \begin{pmatrix} -a(bx_2 + cx_1) & \Omega(x_1, x_2, x_3, a, b, c) \\ 0 & \frac{1}{a}(bx_2 + cx_1) \end{pmatrix}$$

where

$$\begin{aligned} \Omega(x_1, x_2, x_3, a, b, c) &= b^2 x_2 + ic^2 x_1 + 2x_3 a(c - ib) \\ &\quad + \left( bc - b^2 - c^2 - \frac{1}{a^2} \right) (ix_1 + x_2) \end{aligned}$$

and those associated with the generators  $\{E, (iE), H\}$  are

$$\begin{aligned}
g_+^E &= \begin{pmatrix} \frac{1}{2}(\alpha\beta + \bar{\beta}\bar{\alpha})\alpha - \bar{\beta} & \frac{1}{2}(\alpha\beta + \bar{\beta}\bar{\alpha})\beta \\ -\frac{1}{2}(\alpha\beta + \bar{\beta}\bar{\alpha})\bar{\beta} & \frac{1}{2}(\alpha\beta + \bar{\beta}\bar{\alpha})\bar{\alpha} - \beta \end{pmatrix} \\
g_+^{(iE)} &= \begin{pmatrix} -\frac{1}{2}(i\alpha\beta - i\bar{\beta}\bar{\alpha})\alpha - i\bar{\beta} & -\frac{1}{2}(i\alpha\beta - i\bar{\beta}\bar{\alpha})\beta \\ \frac{1}{2}(i\alpha\beta - i\bar{\beta}\bar{\alpha})\bar{\beta} & -\frac{1}{2}(i\alpha\beta - i\bar{\beta}\bar{\alpha})\bar{\alpha} + i\beta \end{pmatrix} \\
g_+^H &= 2 \begin{pmatrix} |\beta|^2\alpha & -|\alpha|^2\beta \\ |\alpha|^2\bar{\beta} & |\beta|^2\bar{\alpha} \end{pmatrix}
\end{aligned}$$

### 5.3.1 Semidirect product with the $\tau$ -action action

Now, we shall work with  $G = SL(2, \mathbb{C})$  and its Lie algebra  $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$  embodied in the semidirect product Lie group  $H = SL(2, \mathbb{C}) \ltimes \mathfrak{sl}_2(\mathbb{C})$  where the vector space  $\mathfrak{sl}_2(\mathbb{C})$  regarded as the representation space for the  $\tau$ -action in the right action structure of semidirect product (5). The associated Lie algebra is the semidirect sum Lie algebra  $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{C}) \ltimes \mathfrak{sl}_2(\mathbb{C})$ . It inherit a decomposition from that in  $\mathfrak{sl}_2(\mathbb{C})$ , namely,  $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$  with

$$\begin{aligned}
\mathfrak{h}_+ &= \mathfrak{g}_+ \oplus \mathfrak{g}_+^\perp = \mathfrak{su}_2 \oplus \mathfrak{b} \\
\mathfrak{h}_- &= \mathfrak{g}_- \oplus \mathfrak{g}_-^\perp = \mathfrak{b} \oplus \mathfrak{su}_2
\end{aligned}$$

The bilinear form

$$((X, X'), (Y, Y'))_{\mathfrak{h}} = g(X, Y') + g(Y, X')$$

turns  $\mathfrak{h}_+, \mathfrak{h}_-$  into isotropic subspaces of  $\mathfrak{h}$ . The Lie group  $H = SL(\mathbb{C}) \ltimes \mathfrak{sl}_2(\mathbb{C})$  factorize also as  $H = H_+ H_-$  with

$$H_+ = SU(2) \ltimes \mathfrak{b} \quad , \quad H_- = B \ltimes \mathfrak{su}_2$$

Then, a typical element  $h \in H$  can be written as

$$h = (g, X) = \left(g_+, \tau_{g_-} \Pi_{\mathfrak{g}_+^\perp} X\right) \cdot \left(g_-, \Pi_{\mathfrak{g}_+} X\right)$$



Let us determine de PL bivector on  $H_+$  from relation

$$\begin{aligned} & \left\langle \gamma(X'_-, Y'^\perp) (g_+, Y_+^\perp)^{-1} \otimes \gamma(X''_-, Y''^\perp) (g_+, Y_+^\perp)^{-1}, \pi_+(g_+, Z_+^\perp) \right\rangle \\ : &= \left( \Pi_- \text{Ad}_{(g_+, Z_+^\perp)^{-1}}^H(X'_-, Y'^\perp), \Pi_+ \text{Ad}_{(g_+, Z_+^\perp)^{-1}}^H(X''_-, Y''^\perp) \right)_{\mathfrak{h}} \end{aligned}$$

By writing

$$\begin{aligned} X_- &= x_E E + x_{(iE)}(iE) + x_H H \\ Y_-^\perp &= y_1 X_1 + y_2 X_2 + y_3 X_3 \\ Z_+^\perp &= z_E E + z_{(iE)}(iE) + z_H H \end{aligned}$$

and introducing  $\pi_+^R : H_+ \longrightarrow \mathfrak{h}_+ \wedge \mathfrak{h}_+$  such that

$$\begin{aligned} & \left\langle \gamma(X'_-, Y'^\perp) (g_+, Y_+^\perp)^{-1} \otimes \gamma(X''_-, Y''^\perp) (g_+, Y_+^\perp)^{-1}, \pi_+(g_+, Z_+^\perp) \right\rangle \\ &= ((X'_-, Y'^\perp) \otimes (X''_-, Y''^\perp), \pi_+^R(g_+, Z_+^\perp))_{\mathfrak{h}} \end{aligned}$$

we obtain

$$\begin{aligned} & \pi_+^R(g_+(\alpha, \beta), z_E E + z_{(iE)}(iE) + z_H H) \\ = & |\beta|^2 (X_1, 0) \wedge (0, (iE)) \\ & - \left( (|\alpha|^2 - |\beta|^2) \text{Im}(\beta\alpha) + 2i\text{Re}(\beta^3 \bar{\alpha}) \right) (X_1, 0) \wedge (0, H) \\ & - \left( 2\text{Re}(\alpha^2 \beta^2) + (|\alpha|^2 - |\beta|^2) |\beta|^2 \right) (X_2, 0) \wedge (0, E) \\ & - 2\text{Im}(\alpha^2 \beta^2) (X_2, 0) \wedge (0, (iE)) \\ & + \left( (|\alpha|^2 - |\beta|^2) \text{Re}(\alpha\beta) + 2\text{Re}(\beta^3 \bar{\alpha}) \right) (X_2, 0) \wedge (0, H) \\ & - \text{Re}(\alpha\beta) (X_3, 0) \wedge (0, (iE)) - 2\text{Im}(\alpha^2 \bar{\beta}^2) (X_3, 0) \wedge (0, H) \\ & - \left( z_E \left( (1 - 4|\beta|^2) \text{Re}(\alpha^2) + (1 - 4|\alpha|^2) \text{Re}(\beta^2) \right) \right) (0, E) \wedge (0, H) \\ & + z_{(iE)} \text{Im}(\alpha^2 - \beta^2) (0, E) \wedge (0, H) \\ & + \left( z_E \left( (1 - 4|\beta|^2) \text{Im}(\alpha^2) + (1 - 4|\alpha|^2) \text{Im}(\beta^2) \right) \right) (0, H) \wedge (0, (iE)) \\ & + z_{(iE)} \text{Re}(\alpha^2 - \beta^2) (0, H) \wedge (0, (iE)) \\ & + 2(z_E \text{Im}(\alpha\bar{\beta}) + z_{(iE)} \text{Re}(\alpha\bar{\beta})) (0, (iE)) \wedge (0, E) \end{aligned}$$

On the side, the Poisson-Lie bivector on  $H_-$  is defined from relation

$$\begin{aligned} & \left\langle (g_-, Z_-^\perp)^{-1} \gamma(X'_+, Y'^\perp_+) \otimes (g_-, Z_-^\perp)^{-1} \gamma(X''_+, Y''^\perp_+), \pi_- (g_-, Z_-^\perp) \right\rangle \\ : &= \left( \Pi_+ \text{Ad}_{(g_-, Z_-^\perp)}^H (X'_+, Y'^\perp_+), \Pi_- \text{Ad}_{(g_-, Z_-^\perp)}^H (X''_+, Y''^\perp_+) \right)_{\mathfrak{h}} \end{aligned}$$

that writing

$$\begin{aligned} X_+ &= x_1 X_1 + x_2 X_2 + x_3 X_3 \\ Y_+^\perp &= b_1 B_1 + b_2 B_2 + b_3 B_3 \end{aligned}$$

and introducing  $\pi_H^- : H_- \longrightarrow \mathfrak{h}_- \wedge \mathfrak{h}_-$  such that

$$\begin{aligned} & \left\langle (g_-, Z_-^\perp)^{-1} \gamma(X'_+, Y'^\perp_+) \otimes (g_-, Z_-^\perp)^{-1} \gamma(X''_+, Y''^\perp_+), \pi_- (g_-, Z_-^\perp) \right\rangle \\ = & \left\langle (g_-, Z_-^\perp)^{-1} \gamma(X'_+, Y'^\perp_+) \otimes (g_-, Z_-^\perp)^{-1} \gamma(X''_+, Y''^\perp_+), \pi_- (g_-, Z_-^\perp) \right\rangle \end{aligned}$$

we get

$$\begin{aligned} & \pi_-^L (g_-, Z_-^\perp) \\ = & \frac{1}{a^2} \left( c^2 + b^2 + \frac{1}{a^2} - a^2 \right) (B_1, 0) \wedge (0, X_2) \\ & - \frac{1}{a^2} \left( b^2 + c^2 + \frac{1}{a^2} - a^2 \right) (B_2, 0) \wedge (0, X_1) \\ & + \frac{c}{a} (B_1, 0) \wedge (0, X_3) - \frac{c}{a} (B_3, 0) \wedge (0, X_1) \\ & - \frac{b}{a} (B_2, 0) \wedge (0, X_3) - \frac{b}{a} (B_3, 0) \wedge (0, X_2) \\ & + 2 \frac{1}{a^2} \left( bc z_1 - ac z_2 + 2 \left( c^2 + b^2 + \frac{1}{a^2} \right) z_3 \right) (0, X_1) \wedge (0, X_2) \\ & + \frac{1}{a} (a z_1 - 2b z_3) (0, X_2) \wedge (0, X_3) - \frac{1}{a} (a z_2 + 2c z_3) (0, X_1) \wedge (0, X_3) \end{aligned}$$

### 5.3.2 AKS integrable system on $\mathfrak{h}_-$

We study a Hamiltonian system on  $\mathfrak{h}_-$ , described by the  $\text{Ad}^H$ -invariant function  $H$ , and for which the hamiltonian vector field is

$$V_H (X_- + K_+) = \Pi_- \text{ad}_{\Pi_+ \mathfrak{L}_H(X_- + K_+)}^{\mathfrak{h}} (X_- + K_+) = -\text{ad}_{\Pi_- \mathfrak{L}_H(X_- + K_+)}^{\mathfrak{h}} (X_- + K_+)$$

It is worth to recall that  $\psi(K_+)$  must be a *character* of  $\mathfrak{h}_-$ , which in this case means that

$$K_+ = (k_3 X_3, k_H H)$$

for arbitrary  $k_3, k_H \in \mathbb{R}$ . Thus the Hamilton equation of motion on  $\iota_{K_+}(\mathfrak{h}_-) \subset \mathfrak{h}$  are then

$$\dot{Z}(t) = V_H(Z(t))$$

for the initial condition  $Z(0) = X_{-\circ} + K_+$ .

In particular, we consider the  $\text{Ad}^H$ -invariant Hamilton function

$$H(X, X') = g(X, X')$$

where  $\psi(\Pi_+(X, X')) \in \text{char } \mathfrak{h}_-$ , so that  $(X, X') = (X_- + k_3 X_3, X_-^\perp + k_H H)$ .

Then, the Hamilton function reduces to

$$H(x_E E + x_{(iE)}(iE) + x_H H + k_3 X_3, x_1 X_1 + x_2 X_2 + x_3 X_3 + k_H H) = x_H k_H + x_3 k_3$$

and

$$\mathfrak{L}_H(X, X') = (x_H H + k_3 X_3, k_H H + x_3 X_3)$$

The hamiltonian vector field is

$$\begin{aligned} & V_H(x_E E + x_{(iE)}(iE) + x_H H + k_3 X_3, x_1 X_1 + x_2 X_2 + x_3 X_3 + k_H H) \\ &= (-2x_H(x_E E + x_{(iE)}(iE)), -2(x_E x_3 + x_H x_1)X_1 + 2(x_{(iE)}x_3 - x_H x_2)X_2) \end{aligned}$$

Because  $x_H$  and  $x_3$  are time independent, we write them as

$$x_H = \frac{\alpha}{2}, \quad x_3 = \frac{\beta}{2}$$

and the nontrivial Hamilton equations are

$$\begin{cases} \dot{x}_E = -\alpha x_E \\ \dot{x}_{(iE)} = -\alpha x_{(iE)} \\ \dot{x}_1 = -\beta x_E - \alpha x_1 \\ \dot{x}_2 = \beta x_{(iE)} - \alpha x_2 \end{cases} \quad (28)$$

The solution  $Z(t) = X_-(t) + K_+$  to this set of Hamilton equation is

$$Z(t) = -\text{Ad}_{h_-(t)}^H(X_{-\circ} + K_+)$$

with

$$h_-(t) = (g_-(t), tx_{30}X_3)$$

being such that

$$(t\mathfrak{L}_H(X_{-\circ} + K_+)) = h_+(t) h_-(t)$$

So, in order to solve our problem, we deal with the exponential curve

$$\text{Exp } t(x_{H0}H + k_{30}X_3, k_{H0}H + x_{30}X_3) = \left( e^{t(x_{H0}H + k_{30}X_3)}, tk_{H0}H + tx_{30}X_3 \right)$$

which factorize as

$$\left( e^{t(x_{H0}H + k_{30}X_3)}, tk_{H0}H + tx_{30}X_3 \right) = (g_+(t), tk_{H0}H) \cdot (g_-(t), tx_{30}X_3)$$

The evolution of the system is then driven by the orbit of the curve  $t \mapsto h_-(t) = (g_-(t), tx_{30}X_3)$  where

$$g_-(t) = \begin{pmatrix} e^{tx_{H0}} & 0 \\ 0 & e^{-tx_{H0}} \end{pmatrix}$$

Writing

$$\begin{aligned} Z(t) &= (x_E(t)E + x_{(iE)}(t)(iE) + x_H(t)H + k_3X_3, \\ &\quad x_1(t)X_1 + x_2(t)X_2 + x_3(t)X_3 + k_HH) \end{aligned}$$

and

$$Z(t_0) = (x_{E0}E + x_{(iE)0}(iE) + x_{H0}H + k_3X_3, x_{10}X_1 + x_{20}X_2 + x_{30}X_3 + k_HH)$$

we have, after some computation and introducing

$$x_{H0} = \frac{\alpha}{2} \quad , \quad x_{30} = \frac{\beta}{2}$$

it turns in

$$\begin{aligned} Z(t) &= - \left( x_{E0}e^{\alpha t}E + x_{(iE)0}e^{\alpha t}(iE) + \frac{\alpha}{2}H + k_3X_3, \right. \\ &\quad \left. (x_{10} + \beta x_{E0}t)e^{\alpha t}X_1 + (x_{20} - \beta x_{(iE)0}t)e^{\alpha t}X_2 + \frac{\beta}{2}X_3 + k_HH \right) \end{aligned}$$

which means that

$$\begin{cases} x_E(t) = -x_{E0}e^{\alpha t} \\ x_{(iE)}(t) = -x_{(iE)0}e^{\alpha t} \\ x_1(t) = -x_{10}e^{\alpha t} - \beta x_{E0}te^{\alpha t} \\ x_2(t) = -x_{20}e^{\alpha t} + \beta x_{(iE)0}te^{\alpha t} \end{cases}$$

that solves the equations (28).

## 6 Conclusions

In this work we developed a method to obtain AKS integrable systems overtaking the lack of an Ad-invariant bilinear form in the associated Lie algebra. In doing so, we promoted the original double Lie algebra to the framework of the semidirect product with the  $\tau$ -representation, obtaining a double semidirect product Lie algebra which naturally admits an Ad-invariant bilinear form. Actually, the structure is richer than this since it amounts to *Manin triple*. So, the method not only brings the problem to the very realm of AKS theory, but also to that of Lie bialgebras and Poisson-Lie groups.

We developed these issues in extent, showing how the AKS theory works in the proposed framework producing Lax pairs equations, and that by freezing the first coordinate in the semidirect product, the  $\tau$ -orbits formulation is retrieved. We built the crossed dressing actions of the factors in the semidirect product Lie group, and showed that the dynamical systems moves on the associated orbits. The Lie bialgebra and Poisson-Lie structures were obtained, however we do not advanced forward the coboundary ones leaving them a pending problem.

In the examples we explored three different situations lacking of Ad-invariant bilinear form: a nilpotent and a solvable Lie algebras, and  $\mathfrak{sl}_2(\mathbb{C})$  where the usual Ad-invariant bilinear form is spoiled with the aim of getting an inner product which gives rise to a Riemannian metric on  $SL(2, \mathbb{C})$ .

In summary, it has been shown that an AKS scheme and Lie bialgebra - PL structures can be associated to any Lie algebra without Ad-invariant bilinear form by promoting them to the semidirect product with the  $\tau$ -representation.

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## References

- [1] M. Adler, P. van Moerbeke, *Completely integrable systems, Euclidean Lie algebras and curves*, Adv. Math. 38 (1980), 267-317.
- [2] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York (1989).
- [3] S. Capriotti, H. Montani, *Integrable systems on semidirect product Lie groups*, J. Phys. A, in press. arXiv: math-ph/1307.0122.
- [4] R. Ghanam, F. Hindelah, and G. Thompson, *Bi-invariant and noninvariant metrics on Lie groups*, J. Math. Phys. 48(2007), 102903-17.
- [5] V. Guillemin, S. Sternberg, *Symplectic techniques in physics*, Cambridge, Cambridge Univ. Press, 1984.
- [6] B. Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. Math. 34 (1979), 195-338.
- [7] J.-H. Lu, A. Weinstein, *Poisson Lie groups, dressing transformations and Bruhat decompositions*, J. Diff. Geom. 31 (1990), 501-526.
- [8] J. Marsden, A. Weinstein, T. Ratiu, *Semidirect products and reduction in mechanics*, Trans. AMS 281 (1984), 147-177; *Reduction and hamiltonian structures on duals of semidirect product Lie algebras*, Contemporary Math. 28 (1984), 55-100.
- [9] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Advances in Math. 21 (1976), 293-329.
- [10] G. Ovando, *Hamiltonian systems related to invariant metrics*, Journal of Physics: Conference Series 175 (2009) 012012-11.
- [11] G. Ovando, *Invariant metrics and Hamiltonian systems*, arXiv:math/0301332v2 [math.DG] 8 Mar 2003.

- [12] M.A. Semenov-Tian-Shansky, *Dressing transformations and Poisson group actions*, Publ. RIMS, Kyoto Univ. 21 (1985), 1237-1260.
- [13] W. Symes, *Systems of Toda type, inverse spectral problem and representation theory*, Inv. Math 159 (1980), 13-51.
- [14] V.V. Trofimov, *Extensions of Lie algebras and Hamiltonian systems*, Math. USSR Izvestiya 23 (1984), 561-578.